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Complex-Plane Strategy: Application to Astrophysical Problems

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Abstract:
The complex-plane strategy is an efficient alternative for treating initial value problems in ordinary differential equations with: (i) singularities and/or indeterminate forms on the real axis, and (ii) terms that become undefined on the real axis when the independent variable exceeds a particular value. The alternative proposed by the complex-plane strategy is to integrate numerically the ordinary differential equations of the problem in the complex plane, and, if necessary due to the appearance of undefined terms on the real axis as in the case (ii) above, along specified contours (i.e. complex paths). We give a brief review on the application of the method to astrophysical problems.

1 Introduction

The "complex-plane strategy" (CPS) proposes and applies numerical integration of "ordinary differential equations" (ODE, ODEs) in the complex plane, either along an interval $\mathbb{I} \subset \mathbb{R}$ when the independent variable r is real, $r \in \mathbb{I}$, or along a "contour" (i.e. "complex path") $\mathfrak{C} \subset \mathbb{C}$ when r is complex, $r \in \mathfrak{C}$. The necessity for integrating in $\mathbb{C}$ arises when the "initial value problem" (IVP, IVPs) is defined on ODEs which: (i) suffer from singularities and/or indeterminate forms on $\mathbb{R}$ (usually at $r = 0$ and/or at $r = R$; typically, R is the radius of the model and has to be computed by the method too), and/or (ii) involve terms that become undefined on $\mathbb{R}$ when the independent variable r exceeds a particular value (usually, when $r > R$). CPS has introduced to problems of astrophysics the idea of transforming real-valued functions of one real variable, i.e. of the real distance, into complex-valued functions of one complex variable, i.e. of the "complex distance" ([1, Sec. 3]), in order to avoid the pathologies described above. Thus the complex path $\mathfrak{C}$, along which integration proceeds, is a "complex distance detour". A similar idea has been applied to problems of dynamics with the time playing the role of the independent variable; whence the titles used in such articles are "complex time integration ...", "complex time detour ...", etc ([2, and references therein]).

2 Outline of the Method

CPS has been applied to astrophysical problems in which the well-known polytropic "equation of state" (EOS) is involved, or other EOSs with similar mathematical characteristics. Such problems are the classical polytropic models ([3, and references therein]); white dwarf models obeying Chandrasekhar's EOS ([4, and references therein]); the solar and jovian systems ([5], [1], [6]); and the general-relativistic polytropic models simulating neutron stars ([7], [8]). This method extends numerical integration of the differential equations involved in the IVP well beyond the radius R of the nonrotating star instead of terminating these integrations just below R. Thus CPS knows the distortion caused by rotation and/or magnetic field over a sufficiently extended space surrounding the initially spherical configuration. So, to compute a particular rotating and/or magnetic model, CPS does not extrapolate beyond the end of the function tables constructed by such extended numerical integrations. It is exactly the avoidance of any extrapolation which keeps the error in the computations appreciably small (it is worth mentioning here that, in numerical analysis, interpolation is a safe and accurate procedure, while extrapolation suffers from large errors and, in most cases, becomes unreliable).

To point out some interesting aspects of CPS, we consider the Lane–Emden differential equation governing the classical polytropic models,

$$\frac{d^2\,\theta}{d\,\xi} + \frac{2}{\xi}\,\frac{d\,\theta}{d\,\xi} = -\theta^n, \qquad \theta(\xi=0)=1, \qquad \theta'(\xi=0)=0. \tag{1}$$

The polytropic index n lies in the interval $n \in \mathbb{I}_n = [0,5] \subset \mathbb{R}$; thus n is a real model parameter. The Lane–Emden function $\theta(\xi)$ becomes zero at its first root ξ_1, $\theta(\xi_1)=0$, and then changes into negative, $\theta(\xi > \xi_1) < 0$. Consequently, the term θ^n becomes undefined beyond ξ_1 (since raising a negative real number to a real power is not defined on $\mathbb{R}$). We can remove this pathology by defining θ as a complex-valued function, $\theta \in \mathbb{C}$, in one real variable, $\xi \in \mathbb{R}$. However, the "raised-to-real-power" term θ^n suffers from a "non-monodromy pathology" in the sense that exponential and logarithmic functions are involved in the series representation of this term (see e.g. [9, Sec. 28]). To remove the latter pathology, we proceed with a systematic transformation of the raised-to-real-power term(s) by defining θ in terms of an "auxiliary Lane–Emden function" η,

$$\theta = \eta^N, \qquad \theta' = N\,\eta^{(N-1)}\,\eta', \tag{2}$$

where the integer N is taken to be sufficiently large in order for the term $\theta^n = \eta^{Nn}$ to be transformed to a "raised-to-integer-power" term. After some algebra, we obtain the "modified Lane–Emden differential equation" ([1, Eqs. (3.4)–(3.5)]),

$$\eta'' + \frac{2}{\xi}\,\eta' + \frac{N-1}{\eta}\,(\eta')^2 = -\,\frac{1}{N}\,\eta^{N(n-1)+1}, \qquad \eta(0)=(1,0), \qquad \eta'(0)=(0,0). \tag{3}$$

However, when integrating on $\mathbb{R}$, the numerical procedure is terminated just below the first root ξ_1, since there is a singularity at ξ_1 owing to the term $\eta' = \theta'/(N\,\eta^{N-1}) = \theta'/\left[N\,\theta^{(1-1/N)}\right]$. We can avoid this singularity by assuming that the independent variable ξ is a "complex distance", $\xi \in \mathbb{C}$, and that the integration proceeds along a complex path $\mathfrak{C} \subset \mathbb{C}$ parallel to the real axis and at a relatively small imaginary distance from it, playing the role of a complex distance detour.

In summary, we have seen that even a simple IVP, defined on the Lane–Emden differential equation which involves the undefined term θ^n for $\xi > \xi_1$, needs to be transformed into an IVP of a complex-valued function in one complex variable, in order for this IVP to get free of the pathologies discussed above.

References

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