The influence of strong field vacuum polarization on gravitational - electromagnetic wave interaction

D. Papadopoulos
Department of Physics
Section of Astrophysics, Astronomy and Mechanics

O u t l in e :

We will discuss the interaction between gravitational and electromagnetic waves in the presence of a strong static magnetic ¯eld .

In our model, the ¯eld strength of the static ¯eld is allowed to surpass the Schwinger critical ¯eld, such that the quantum electrodynamical (QED) e®ects of vacuum polarization and magnetization are signi¯cant.

Equations governing the interaction will be derived and analyzed.

Results:

We will see that the energy conversion from gravitational to electromagnetic waves can be significantly altered due to the QED effects.

The consequences of our results will be discussed.

Some words for us:

Conversion of energy from gravitational to electromagentic degrees of free-dom has been examined by several authors as a means to indirect detection of gravitational waves, since the latter is so much easier to detect.

For astrophysical application, naturally this requires well developed theories to recognize the signature of the gravitational origin.

Here we propose a model where the propagation of GWs across an external static magnetic ¯eld gives rise to a linear coupling to the electromagnetic ¯eld, which may lead to the GW excitation of ordinary EM waves in vacuum, or of magnetohydrodynamic (MHD) waves in a plasma.

The Equations:

We Start with Einstein's equations:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = kT_{\mu\nu} \qquad (1a)$$

and Maxwell's equations which in CED are linear and are derived from the Lagrangian density (varying the four-potential).

$$\mathcal{L}_c = -(1/4\mu_0) F^{\alpha\beta} F_{\alpha\beta} \qquad (1b)$$

$F^{\alpha\beta}$ is the electromagnetic field tensor.

When QED enters the picture changes, since the photons may interact in the absence of real charged sources.

Maxwell's equations get new types of source terms that are nonlinear in the field strengths.

Schwinger's Lagrangian:

$$L = -\frac{1}{\mu_0}\mathcal{F} - \frac{\alpha}{2\pi\hbar\mu_0 e^2}\int_0^{\infty}\frac{ds}{s^3}e^{-eE_{cr}s/c}$$

$$\times\left[(es)^2 ab \coth(eas)\cot(ebs) - \frac{(es)^2}{3}(a^2 - b^2) - 1\right] - A_\alpha j^\alpha \qquad (2)$$

where $a = \left[\sqrt{(\mathcal{F}^2 + \mathcal{G}^2)} + \mathcal{F}\right]^{1/2}$, $b = \left[\sqrt{(\mathcal{F}^2 + \mathcal{G}^2)} - \mathcal{F}\right]^{1/2}$

$\mathcal{F} = (1/4)F_{\alpha\beta}F^{\alpha\beta}$, $\mathcal{G} = (1/4)F_{\alpha\beta}\hat{F}^{\alpha\beta}$

$\hat{F}^{\alpha\beta} = \frac{\epsilon^{\alpha\beta\gamma\delta}F_{\gamma\delta}}{2}$,

A_α the four-potential, j^α the four-current and α the fine structure constant.

The Euler-Lagrange equations of Motion:

$$q_F F^{\mu\nu}_{;\mu} + q_G \tilde F^{\mu\nu}_{;\mu} + \frac{1}{2}\left[q_{FF} F^{\mu\nu} F_{\alpha\beta} + q_{GG} \tilde F^{\mu\nu} \tilde F_{\alpha\beta}\right] F^{\alpha\beta}_{;\mu}$$

$$+ q_{FG} \left\langle F^{\mu\nu} \tilde F_{\alpha\beta} + \tilde F^{\mu\nu} F_{\alpha\beta} \right\rangle F^{\alpha\beta}_{;\mu} = -j^{\nu} \qquad (4)$$

where

$$q_F = \frac{\partial L}{\partial F}; \quad q_G = \frac{\partial L}{\partial G}; \quad q_{FF} = \frac{\partial^2 L}{\partial F^2}; \qquad (5a)$$

$$q_{GG} = \frac{\partial^2 L}{\partial G^2}; \quad q_{FG} = \frac{\partial^2 L}{\partial F \partial G} \qquad (5b)$$

The Metric:

Here we will discuss the influence of a GW on a strong magnetic field.

The metric of a linearized GW propagating in the z-direction can be written

$$ds^2 = -c^2dt^2 + (1 + h_+)\,dx^2 + (1 - h_+)\,dy^2 + 2h_\times\,dx\,dy + dz^2 \qquad (6)$$

where the two independent polarizations h_+ and $h_\times$ depend on the coordinates as

$$h_{+,\times} = h_{+,\times}(z - ct) \qquad (7)$$

.

Einstein’s Equations in Tetrad formalism:

In a tetrad given by

$$e_0 = \frac{1}{c}\partial_t ;\ e_1 = \left(1 - \frac{1}{2}h_+\right)\partial_x - \frac{1}{2}h_\times \partial_y ; \qquad (8a)$$

$$e_2 = \left(1 + \frac{1}{2}h_+\right)\partial_y - \frac{1}{2}h_\times \partial_x ;\ e_3 = \partial_z \qquad (8b)$$

Einstein's equations in the linearized theory of gravity read:

$$(e_0^2 - \partial_z^2)h_+ = \kappa\,(dT_{11} - dT_{22}); \qquad (9a)$$

$$(e_0^2 - \partial_z^2)h_\times = 2\kappa\,(dT_{12}) \qquad (9b)$$

where $\kappa = 8\pi G/c^4$, and G is the gravitational constant.

The energy-momentum tensor:

The energy-momentum tensor associated with the Lagrangian is written

$$T_{\mu\nu} = -\theta_F F_{\mu}^{\ \alpha} F_{\alpha\nu} + (G\theta_G - L) g_{\mu\nu} \qquad (10)$$

,

Its perturbed components are linearized around the strong magnetic field B_0 and some of them are:

$$\delta T_{11} = \delta T_{22} = B_0^2 \delta\theta_F - 2\theta_F B_0 \delta B_1 \qquad (10a)$$

$$\delta T_{12} = \theta_F B_0 \delta B_2 \qquad (10b)$$

Next we write Maxwell's Equations in a 1+3 formalism:

Suppose an observer moves with 4-velocity $u^{®}$. This observer will measure the electric and magnetic ¯elds

$$E_{®} = F_{®¯}u^{¯} \qquad (11a)$$

and

$$B_{®} = ²_{®¯q}F^{¯q}=2 \qquad (11b)$$

$F_{®¯}$ is the EM ¯eld tensor

$²_{®¯q}$ is the volume element on hyper-surfaces orthogonal to $u^{®}$.

We also de¯ne the spatial gradient operator as $^{r} = (e_1; e_2; e_3)$.

Maxwell’s Equations in tetrad Formalism:

$$c\,\mathrm{r}\;¢\,B = \frac{½_B}{²_0}\;;\quad \mathrm{r}\;¢\,E = \frac{1}{²_0}\,µ\,\frac{½}{q_F} + ½_E\,¶ \qquad (12a)$$

$$e_0 B + \frac{\mathrm{r}\;£\;E}{c} = i\,{}^1{}_0\,j_B\;;\quad \frac{1}{c}\,e_0 E\;i\;\mathrm{r}\;£\;B = i\,{}^1{}_0\,µ\,j_Q + \frac{j}{q_F} + j_E\,¶ \qquad (12b)$$

where j_Q is

$$j^{®}_{Q} = i\,\frac{1}{2{}^1{}_0}\,µ\,\frac{q_{GG}}{q_F}\,\hat{F}_{kl}\hat{F}_{i®} + \frac{q_{FF}}{q_F}\,F_{kl}F_{i®}\,¶\;e_i F_{kl} \qquad (13)$$

The effective(gravity induced) charge and current densities:

½E = i ²0 q® £ ®E¯ + ²®¯q q0® cB q ¤ (14a)

½B = i ²0 q® £ ®cB¯ i ²®¯q q0® E q ¤ (14b)

j®E = 1 / (1/0) . i (q0® i q®0) E¯/c + q0¯ E®/c i ²®¯q(q0 0 Bq + q· q B.) ¸ (14c)

j®B = 1 / (1/0) . i (q0® i q®0)B¯ + q0¯ B® + ²®¯q µ q0 0 Eq/c + q· q E./c ¶ ¸ (14d)

Summary I:

I. Einstein's Equations :

$$(e_0^2 \text{ i } @_z^2)h_+ = \cdot\,(dT_{11} \text{ i } dT_{22}); \qquad (15a)$$

$$(e_0^2 \text{ i } @_z^2)h_{\pounds} = 2\cdot\,(dT_{12}) \qquad (15b)$$

II. Maxwell's Equations:

$$c\,r\,\text{¢}B = \frac{\text{½}_B}{{}^2_0};\quad r\,\text{¢}E = \frac{1}{{}^2_0}\,\text{µ}\,\frac{\text{½}}{q_F} + \text{½}_E\,\text{¶} \qquad (16a)$$

$$e_0B + \frac{r\,\pounds\,E}{c} = \text{i}\,{}^1{}_0 j_B;\quad \frac{1}{c}e_0E \text{ i } r\,\pounds\,B = \text{i}\,{}^1{}_0\,\text{µ}\,j_Q + \frac{j}{q_F} + j_E\,\text{¶} \qquad (16b)$$

Here our aim is to investigate..

Here our aim is to investigate to what extent the QED effects, associated with fields strengths approaching the Schwinger limit, modifies the energy conversion between GW:s and EM-waves.

For this purpose we make the ansatz:

1. The interaction region is smaller than the radius of curvature due to B_0, such that the interaction can be considered as taking place on a Minkowski background.

2. The metric(Minkowski) perturbations are:

$h_{\times,+} = \tilde{h}_{\times,+} \exp[i(kz - \omega t)]$ with $\omega = kc$,

3. The perturbations to the electromagnetic field are:

$B = B_0 e_1 + dB(z) \exp[-i\omega t]$ and $E = dE(z) \exp[-i\omega t]$,

where dB and dE includes both positive (along z) and negative propagating waves.

Summary II.

I. Einstein's Equations :

$$(e_0^2 - \partial_z^2)h_+ = \kappa\,(dT_{11} - dT_{22}); \quad (e_0^2 - \partial_z^2)h_\times = 2\kappa\,(dT_{12}) \qquad (17)$$

II. Maxwell's Equations:

$$\left[k_E^{+2} + \partial_z^2\right] dB_1 = k_E^{+2} B_0 \tilde{h}_+ e^{ikz}; \quad \left[k_E^{\times 2} + \partial_z^2\right] dB_2 = \left(\frac{1}{2}\frac{\omega^2}{c^2} + k_E^{\times 2}\right) B_0 \tilde{h}_\times e^{ikz} \qquad (18)$$

$$dE_2 = \left(\frac{i}{\omega}\,\frac{\omega^2}{k_E^{+2}}\,\partial_z dB_1 + \frac{B_0}{2}\partial_z h_+\right) \quad dE_1 = -\left(\frac{i\omega}{k_E^{\times 2}}\,\partial_z dB_2 + \frac{B_0}{2}\partial_z h_\times\right) \qquad (20)$$

where $k_E^{+2} = \omega^2 = \left(c^2\left(1 + B_0^2 q_{FF} = q_F\right)\right)$ and $k_E^{\times 2} = \omega^2\left(1 - B_0^2 q_{GG} = q_F\right) = c^2.$

A specific example:

A boundary value problem: Assume that the GW propagates in the e_3-direction and enters in the interaction region, $-L/2 < z < L/2$, which is the region where the external magnetic field $B_0 e_1$ is taken to be nonzero.

The general solution to Eqs. (18), for that region $-L/2 < z < L/2$, is

$$\delta B_{1;2} = T_{1;2} e^{ik_E^{+;\times} z} + R_{1;2} e^{-ik_E^{+;\times} z} + C_{1;2} e^{ikz} \qquad (21)$$

where

$C_1 = k_E^{+2} B_0 \psi_+ /(k_E^{+2} - k^2)$, $C_2 = (k_E^{\times 2} + k^2) B_0 \psi_\times /2(k_E^{\times 2} - k^2)$, and

$R_{1;2}$ and $T_{1;2}$ are constants determined by the boundary conditions.

The matching..

The solution is matched with the EM wave solutions with constant amplitudes outside the interaction region

$$dB_{1;2} = f^R_{1;2} e^{-ikz}; z \in (-\infty; -L/2); dB_{1;2} = f^T_{1;2} e^{ikz}; z \in (L/2; \infty); \quad (22)$$

at $z = -L/2; +L/2$.

The matching of the electric field is done in the same way as that of the magnetic field to give four equations for four quantities, for each set of coupled polarizations.

For the mode that couples to the plus-polarization:

We solve those 4- equations and the resulting amplitudes of the "reflected" and "transmitted" EM-waves are:

$$f_1^R = \frac{B_0 \tilde{h}_+}{2} n_+ e^{-i\theta} \frac{(1+n_+)e^{in_+\theta} + (1-n_+)e^{-in_+\theta} - 2e^{i\theta}}{(1+n_+)^2 e^{-i\theta n_+} - (n_+-1)^2 e^{i\theta n_+}} \qquad (22)$$

and

$$f_1^T = \frac{B_0 \tilde{h}_+}{2} \frac{n_+}{(1-n_+)^2 e^{in_+\theta} - (1+n_+)^2 e^{-in_+\theta}} \times \left[\frac{(1-n_+)^2}{1+n_+} e^{in_+\theta} + \frac{(1+n_+)^2}{(1-n_+)} e^{-in_+\theta} + 2\frac{3n_+^2+1}{n_+^2-1} e^{-i\theta} \right] \qquad (23)$$

For the mode that couples to the cross-polarization:

similarly we obtain

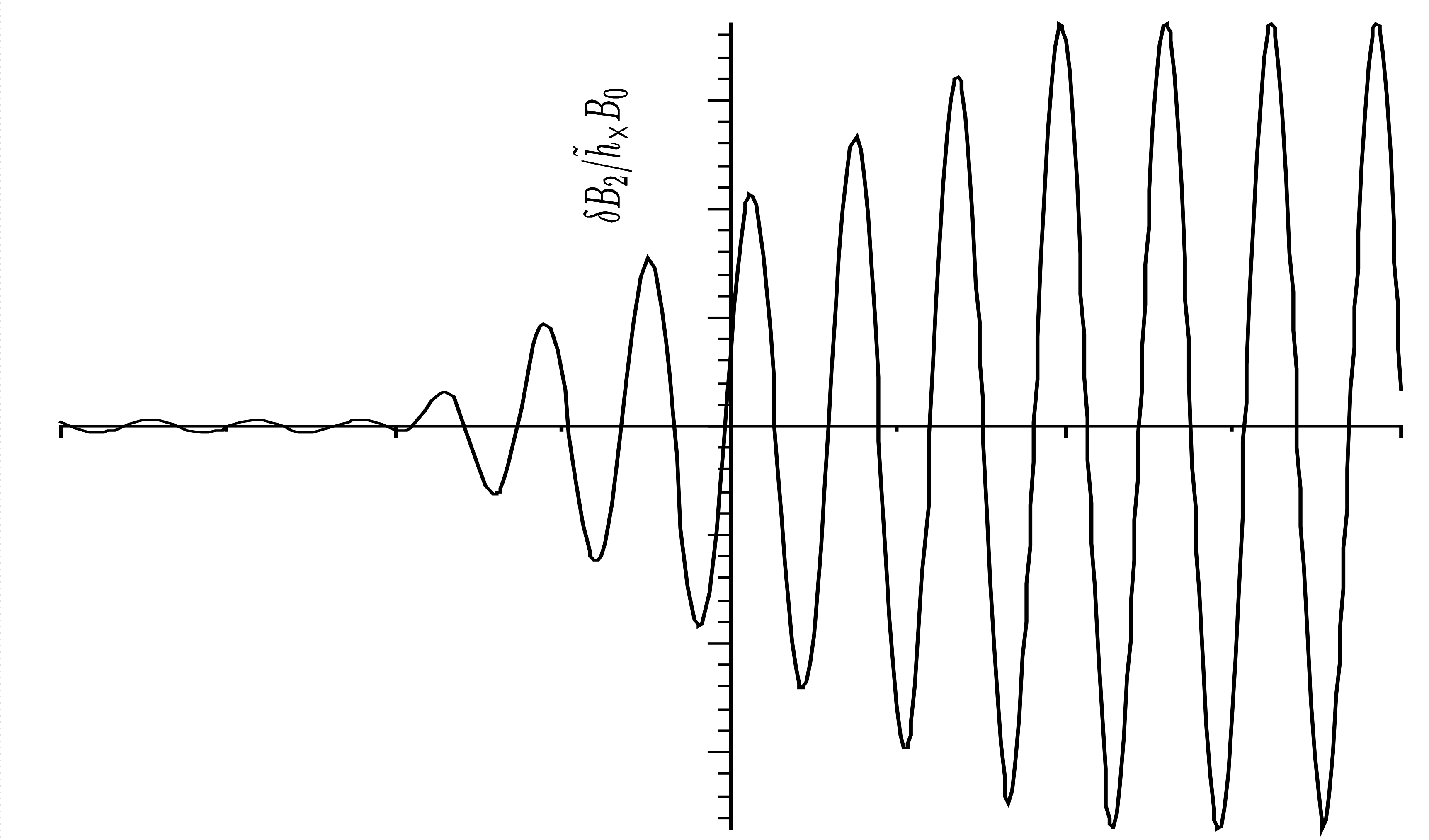
$$f_2^R = \frac{\hbar_{\pounds} B_0}{2} \, \text{i} \, n_{\pounds}^2 + 1 \, \text{¢} \, \frac{\pounds \, e^{\text{i} \, \text{i}\mu} e^{\text{i}\mu n_{\pounds}} \, \text{i} \, e^{\text{i} \, \text{i}\mu n_{\pounds}} \, \text{¤}}{(n_{\pounds} \, \text{i} \, 1)^2 e^{\text{i}\mu n_{\pounds}} \, \text{i} \, (n_{\pounds} + 1)^2 e^{\text{i} \, \text{i}\mu n_{\pounds}}} \quad (24)$$

and

$$f_2^T = \frac{\hbar_{\pounds} B_0}{2} \, \mu \, \frac{n_{\pounds}^2 + 1}{n_{\pounds}^2 \, \text{i} \, 1} \, \P \, \frac{(n_{\pounds} \, \text{i} \, 1)^2 e^{\text{i}\mu n_{\pounds}} \, \text{i} \, (n_{\pounds} + 1)^2 e^{\text{i} \, \text{i}\mu n_{\pounds}} + 4 n_{\pounds} e^{\text{i} \, \text{i}\mu}}{(n_{\pounds} + 1)^2 e^{\text{i} \, \text{i}\mu n_{\pounds}} \, \text{i} \, (n_{\pounds} \, \text{i} \, 1)^2 e^{\text{i}\mu n_{\pounds}}} : \quad (25)$$

where $n_{+;\pounds} = k_E^{+;\pounds} = k$ and $\mu = kL$.

An example of the magnetic pro¯le (containing both the transmitted and re°ected wave) is given in Fig.(1) for $kL = 40$ and $cB_0=E_{cr} = 100$



The $\langle |f_1^T|^2 \rangle$, $\langle |f_2^T|^2 \rangle$ at the limit $n_{+,\times} \to 1$:

The expressions $\langle |f_i^A|^2 \rangle$, $i = 1,2$ and $A = T,R$ contain all information about the energy conversion to the different EM-modes.

However, to appreciate these results and the effects due to QED, we must first evaluate some results for the low-field limit when $n_{+,\times} \to 1$.

At this limit, the $|f_1^T|^2$, for the $+$-polarization is:

$$\langle |f_1^T|^2 \rangle = \frac{1}{4} \langle \tilde{n}_+ \rangle^2 B_0^2 k^2 L^2 \tag{26}$$

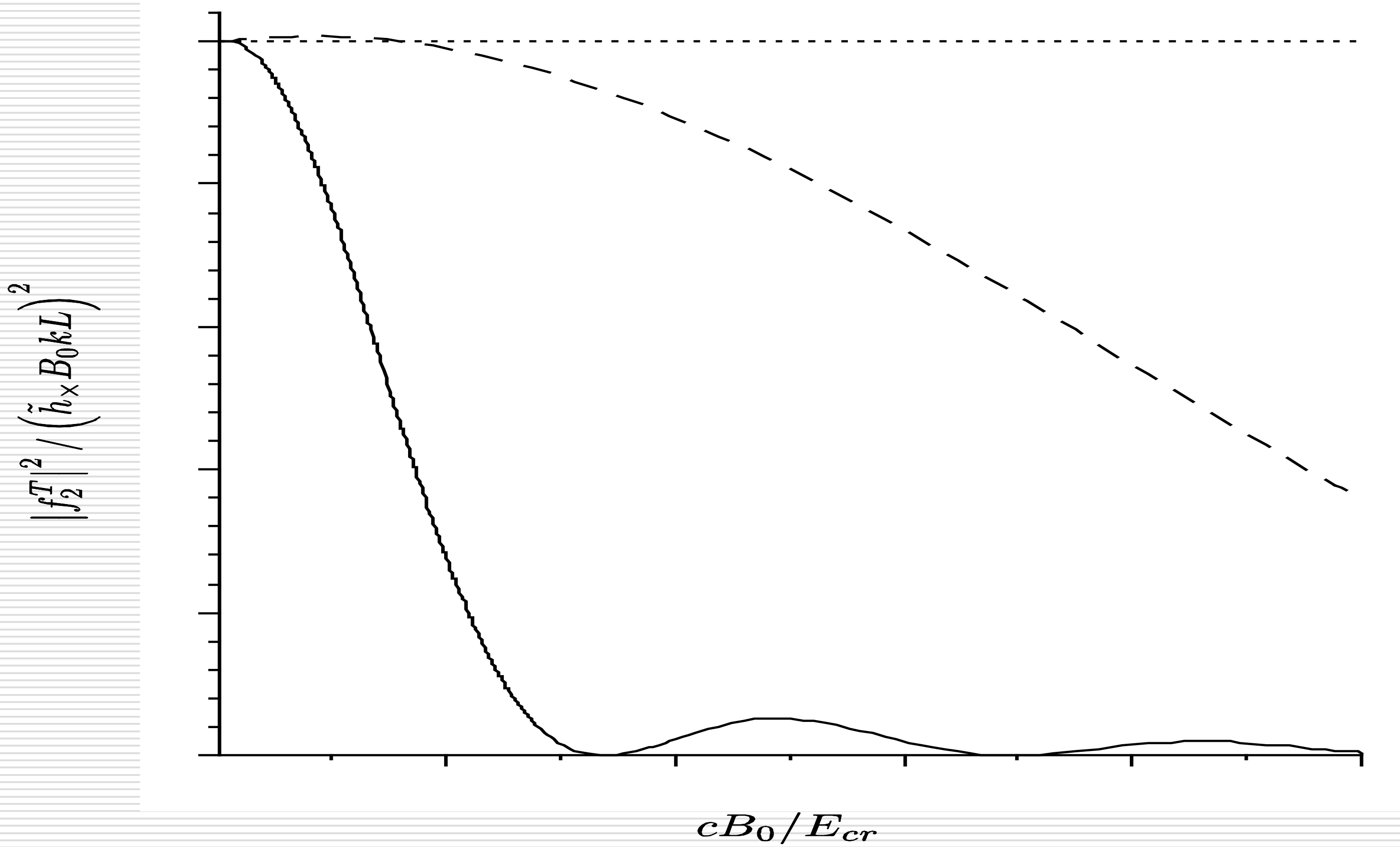
and similarly for the mode excited by the $\times$-polarization is:

$$\langle |f_2^T|^2 \rangle = \frac{1}{4} \langle \tilde{n}_\times \rangle^2 B_0^2 k^2 L^2 \tag{27}$$

.

We see that the transmitted energy density is directly proportional to the background energy density.

Normalizednenergy density of the copropagating-EW Excited by a $h_{\pounds}$ mode :



Comments:

The excited EM-modes change when QED-e®ects are taken into account.

The main reason is that the EM wave dispersion relation is changed in the interaction region (that makes $n_{+,\times}$ deviate from unity) which in turn detunes the excited wave with the GW.

The consequence for the transmitted wave excited by the $\times$-polarization is depicted in Fig.(2), for kL = 20 and kL = 100.

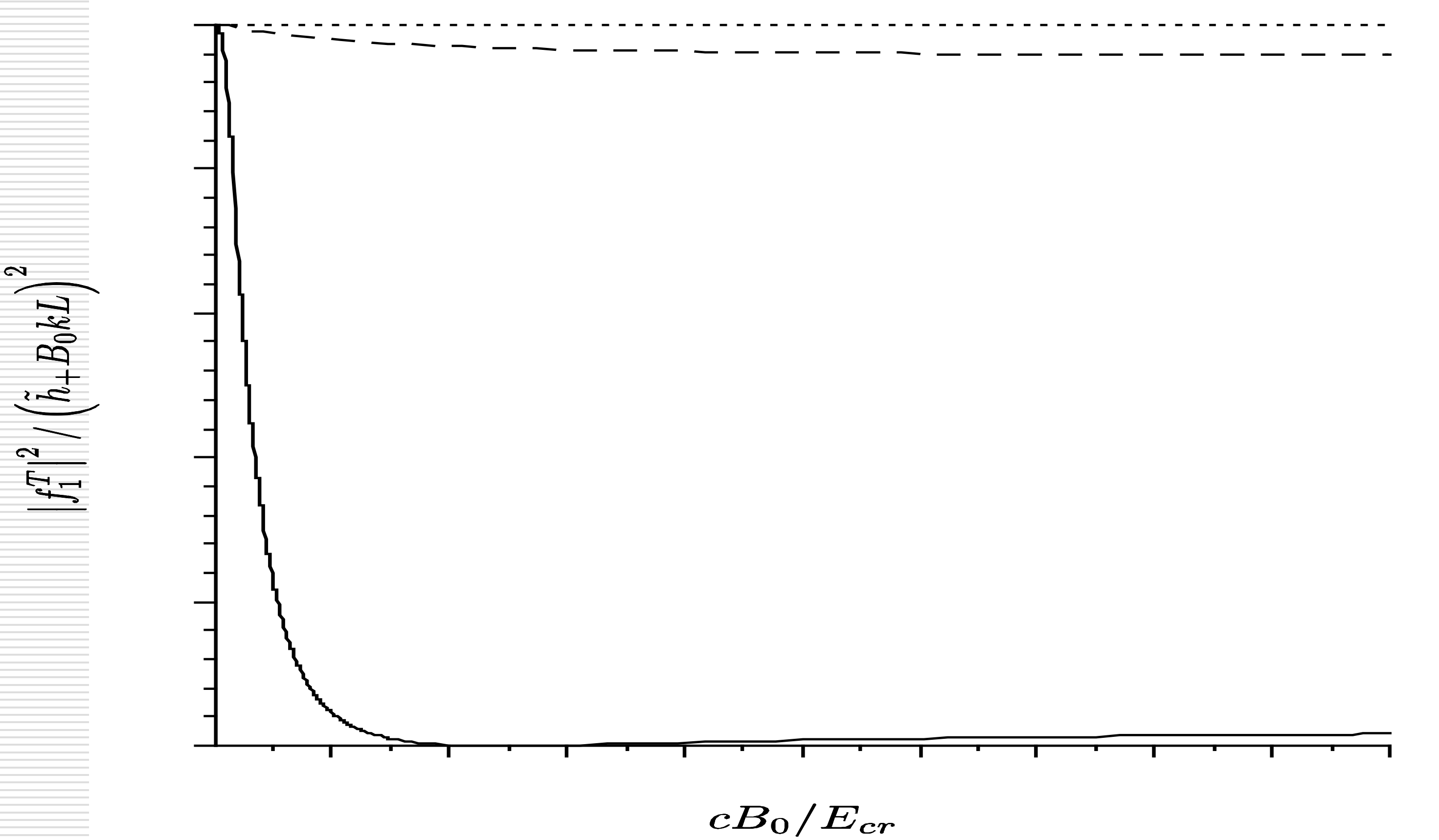
The steady increase in the absence of QED is replaced by an oscillatory behavior, mainly due to the detuning of the GW and EM wave dispersion relation.

Note that we here have normalized the transmission coe±cient with

$$|\tilde{n}_{+,\times}|^2 B_0^2 k^2 L^2,$$

such that the coe±cient without QED-e®ects is represented by a straight line.

Normalized energy density of copropagating EM-wave Excited by h_+ mode:



Comments:

Fig. (3) depicts the energy density for the co-propagating mode excited by the + -polarization.

The energy conversion to this EM-mode is much less a®ected by the QED e®ects.

The reason is that the QED-modi¯cation of the EM dispersion relation e®ectively saturates at a value $cB_0/E_{cr} \approx 10$.

Accordingly we have chosen higher values of kL, namely kL = 2000 and kL = 20000, which is needed in order to see the deviation from the classical behavior induced by QED.

Summary-Conclusions:

1. We studied the interaction between GW:s and EM-waves in the presence of a strong static magnetic ¯eld B_0, using the Heisenberg-Euler Lagrangian to take QED e®ects into account in the high-frequency approximation scheme.

2. The speci¯c boundary conditions considered is an incoming GW incident on a static magnetic ¯eld with a given extent L, which give raise to an excited EM-wave in the same direction as the GW, as well as one propagating in the opposite direction.

3. The role of the QED e®ects is twofold:

a. The coupling strength between the GW:s and the electromagnetic waves are modi¯ed(Einstein's Equations).

b. The change in phase velocity (< c) of the EM-waves induced by the vacuum polarization, as described by the expressions $k^{£}_{E}$ and k^{+}_{E}, destroys the perfect resonance with the GWs, which gives a saturation of the possible energy conservation at a ¯nite value of L.