# Hall equilibria and stability of magnetic field structure in neutron star crusts

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# Outline

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- Hall evolution

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- Equilibria Solutions

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- A stable MHD equilibrium is not necessarily a stable Hall equilibrium

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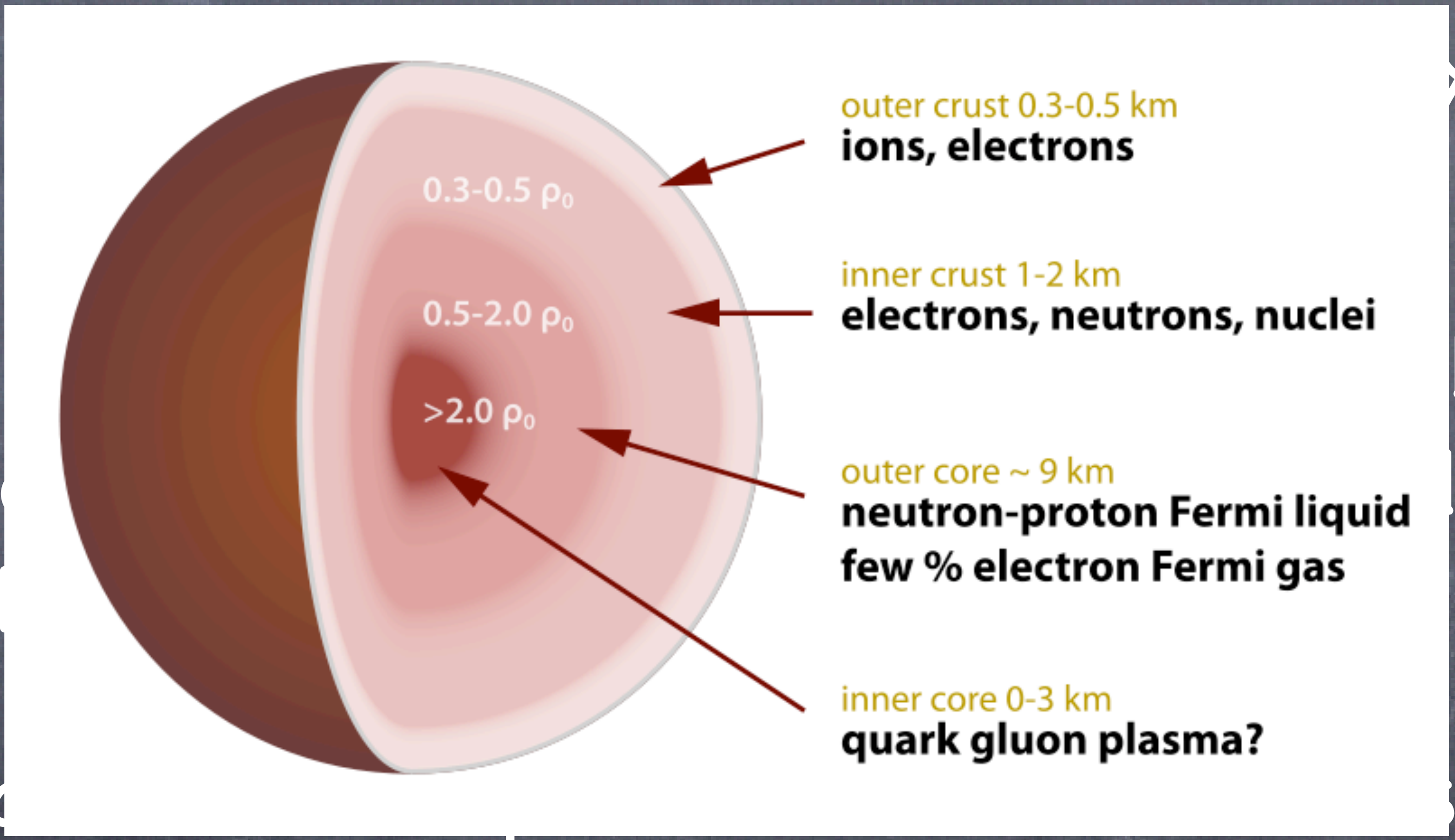

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  - In equilibrium: Lorentz force, pressure gradient and gravity add up to zero
  - A purely poloidal or toroidal field are not stable configurations (Prendergast 1956 and simulations by Braithwaite & Spruit 2006a,b)

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- Here we shall focus on solutions corresponding to purely poloidal fields

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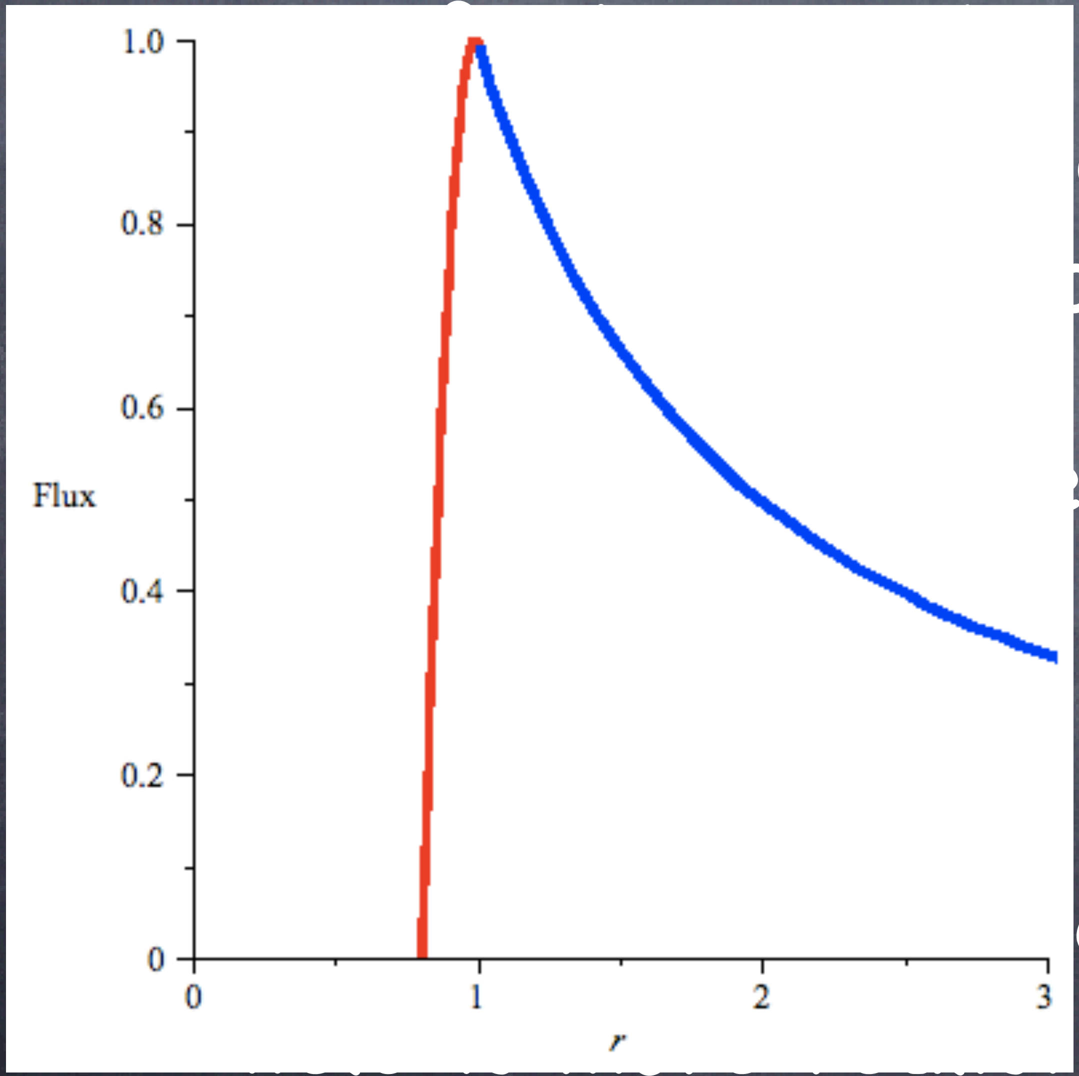
- We are looking for purely poloidal Hall equilibria (a combination of poloidal-toroidal field is more realistic)

# Poloidal equilibria

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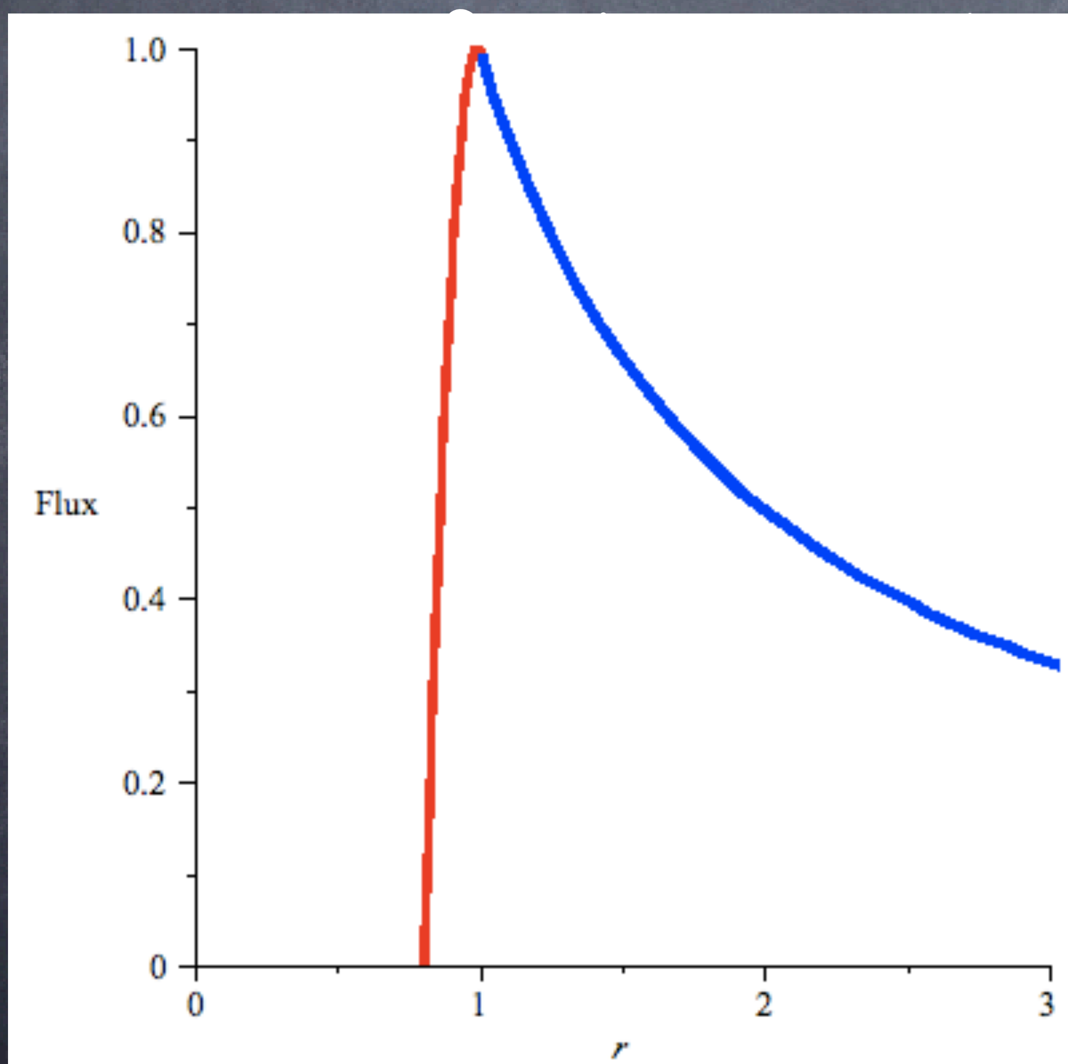


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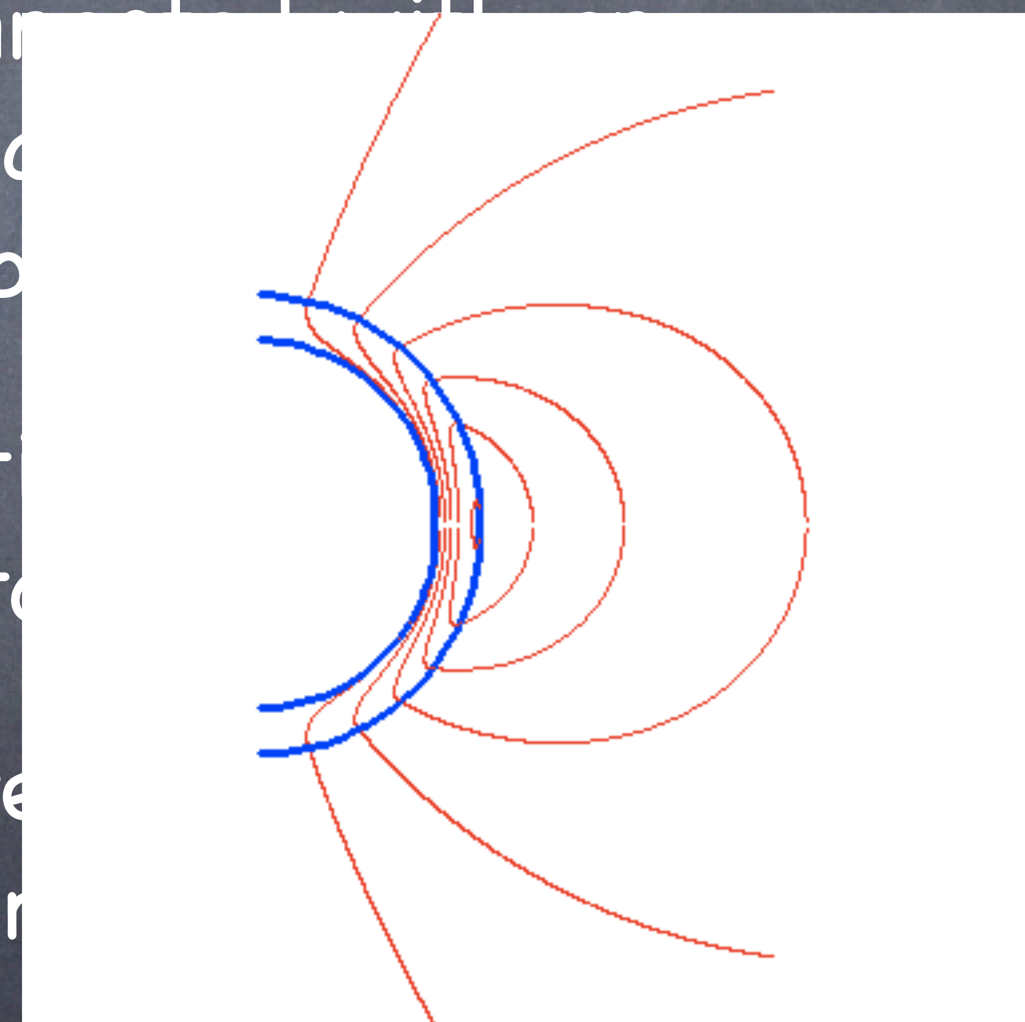

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$$\mathbf{B} = \frac{B_0}{2R^2}[(5R^2 - 3r^2)\cos\theta \mathbf{e}_r - (5R^2 - 6r^2)\sin\theta \mathbf{e}_\theta]$$

$$\mathbf{J} = \frac{15cB_0}{8\pi R^2} r \sin\theta \mathbf{e}_\phi$$

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  - MHD: Dynamical evolution, force equilibrium, variation principle
  - In Hall there are no forces to be accounted for, kinetic evolution

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- In the solution presented the first term is zero, the second can be positive for an appropriate choice of perturbation

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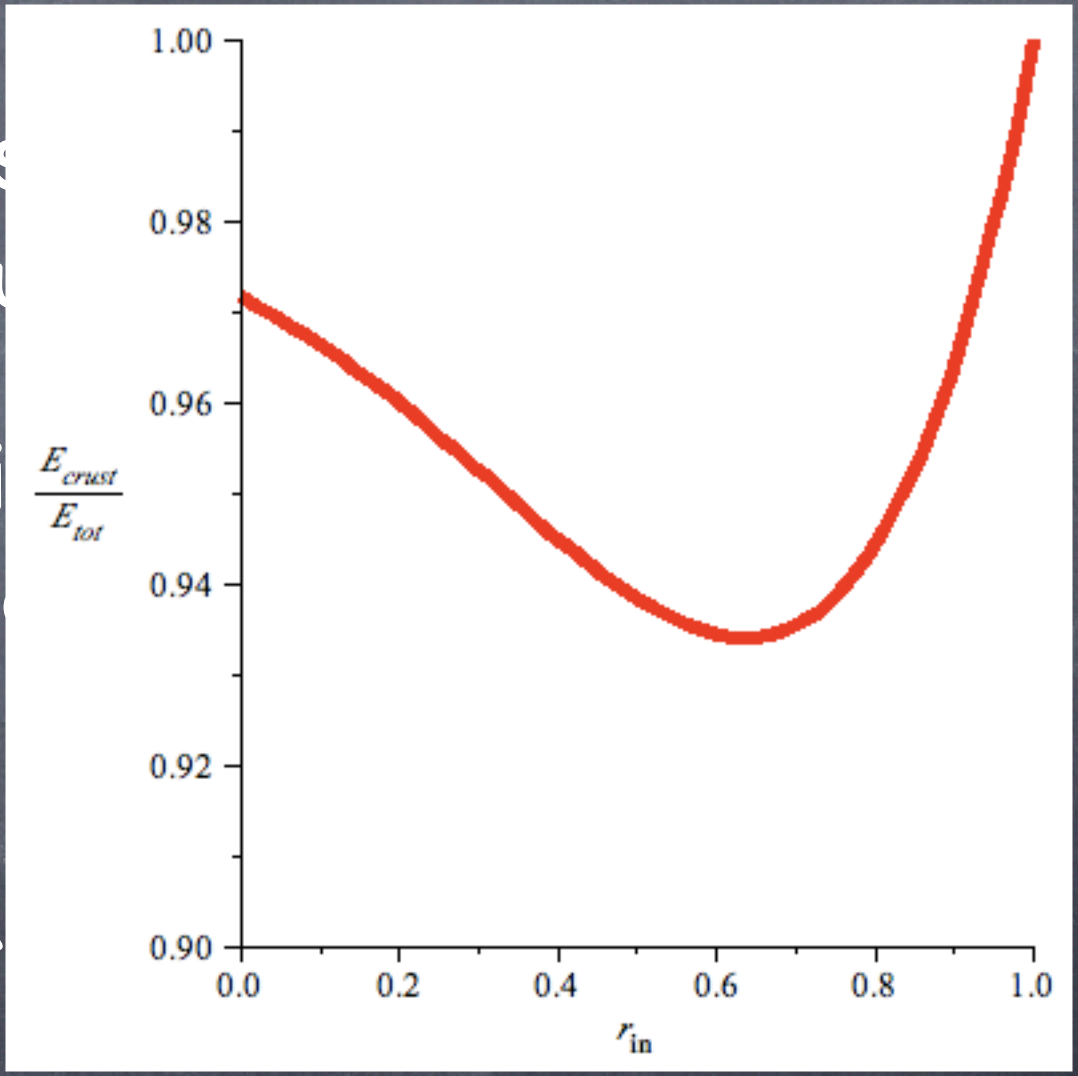

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- The evolution leads towards a stable poloidal field connected to an external dipole
- Magnetar activity may be the outcome of the process of transforming an MHD equilibrium to a Hall equilibrium which loads the magnetosphere with magnetic energy and helicity (i.e. Thompson & Duncan 1993, 1995)

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Thanks!