On the dynamics of a small body under the influence of a Maxwell ring-type N-body system with a spheroidal central primary: Focal curves

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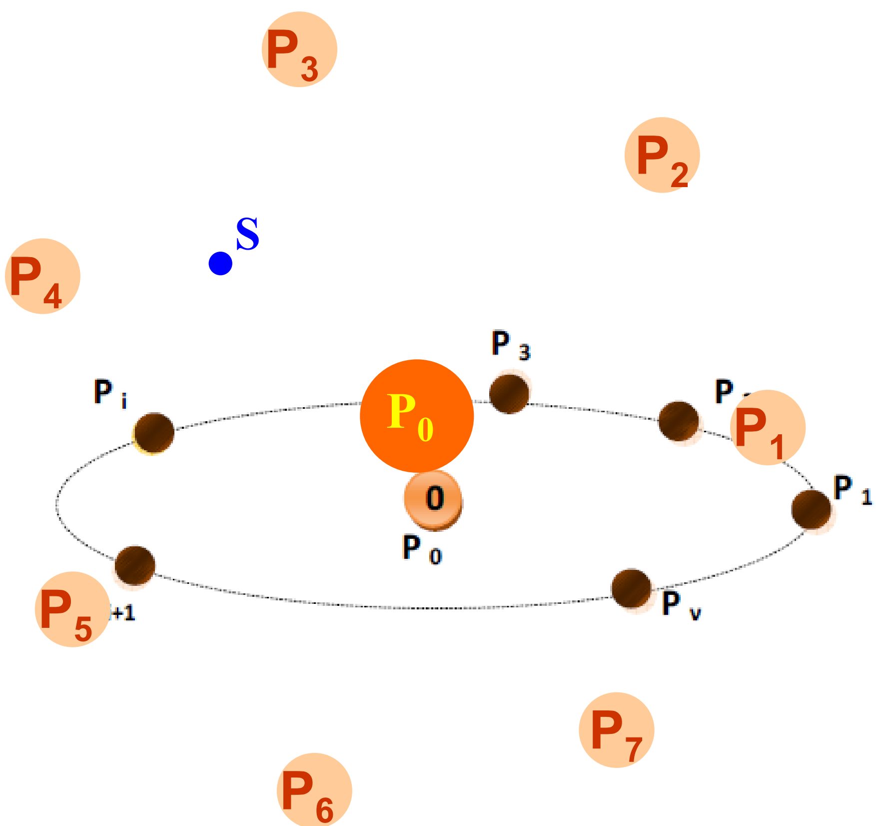
Introduction

Our presentation deals with a new property which has been observed for the first time (Kalvouridis, 2004) in the x-C diagrams during the investigation of the planar motion of a small body in the gravitational force field created by a regular polygon configuration of N bodies.

A brief description of the restricted (N+1)-body regular polygon model (Ring-Problem))



The configuration of the regular polygon problem of (N+1) bodies

- **ν=N-1 of the bodies-members of the system are spherical, homogeneous with equal masses *m*, and are located at the vertices of an imaginary regular ν-gon, while the Nth body has a different mass m_0 and is located at the mass center of the system.**
- **This formation rotates around its mass center with constant angular velocity, so that all the primaries are in relative equilibrium. It has been proved *(Vanderbei, R.J., Kolemen, E. (2007))* that this state may exist when ν>6.**
- **A small body S , natural or artificial, moves under the influence of all the primaries of the system yet having no effect on their motion.**

POST-NEWTONIAN POTENTIALS

The corrective term inserted by Newton in the expression of the law of gravitation

- **In order to explain the motion of the apsidal line of Moon, Newton added a corrective inverse cube term of the form :**

$$-\frac{1}{r^3}$$

to his famous inverse square law of gravitation *(Principia, Book I, Article IX, Proposition XLIV, Theorem XIV, Corollary 2).*

The corrective term proposed by Manev

- **A similar adjustment was proposed by Manev in 1924 in order to explain some relativistic effects without using the theory of relativity, as well as phenomena like the radiation pressure and the oblateness of the bodies.**

$$-\frac{1}{r^2}-\frac{B}{r^3}$$

The improved version of the ring problem where the central body creates a Manev –type potential

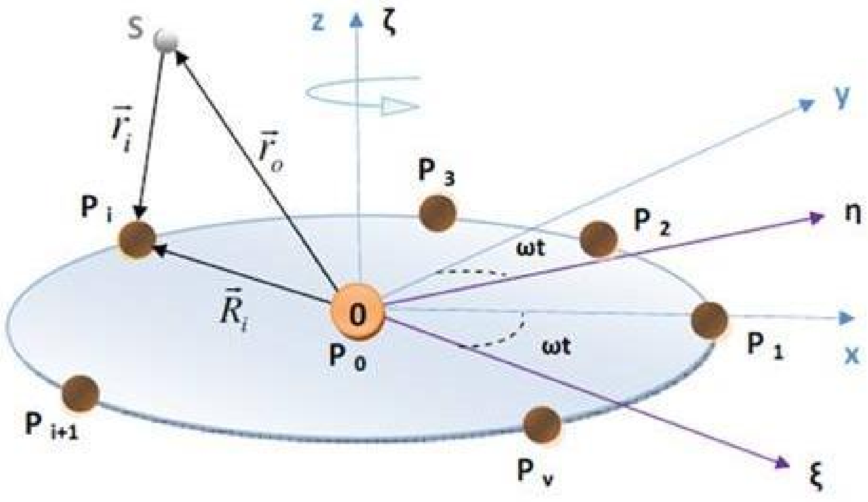
- The ν peripheral bodies create Newtonian force fields
- The central primary creates a Manev-type potential :

$$-\frac{1}{r}-\frac{B}{r^2} \qquad \text{where :} \qquad B = e\,\alpha$$

B is expressed in length units, a is the side of the regular ν-gon and e is a dimensionless coefficient

Parameters of the problem

- Manev's parameter e of the corrective term (small real values)
- number ν of the peripheral primaries (positive integer values)
- mass parameter $\beta=m_0/m$ ($\beta>0$, positive real values)



COORDINATE SYSTEMS

- Inertial coordinate system Oξηζ centered at the mass center O of the primaries' formation.
- Synodic system Oxyz rigidly attached to the primaries which rotates with constant angular velocity, here taken as ω=1.

Dimensionless equations of motion in the synodic system Oxyz

After normalization of the physical quantities

$$\ddot{x} - 2\dot{y} = \frac{\partial U}{\partial x} = U_x, \quad \ddot{y} + 2\dot{x} = \frac{\partial U}{\partial y} = U_y, \quad \ddot{z} = \frac{\partial U}{\partial z} = U_z$$

$$U(x,y,z) = \frac{1}{2}\left(x^2 + y^2\right) + \frac{1}{\Delta}\left[\beta\left(\frac{1}{r_0} + \frac{e}{r^2{}_0}\right) + \sum_{i=1}^{v}\frac{1}{r_i}\right]$$

$$\vec{r}_o = (x, \quad y, \quad z)$$

$$\vec{r}_i = (x - x_i, \quad y - y_i, \quad z)$$

distances of the particle from the central and the peripheral primaries respectively

$$\Delta = M\left(\Lambda + \beta M^2 + 2\beta e M^3\right)$$

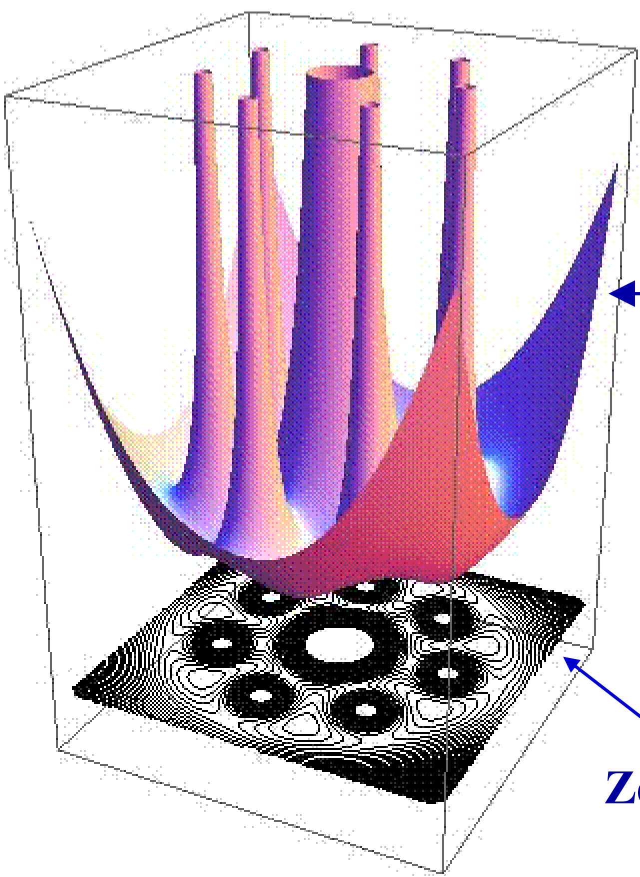
$$\Lambda = \sum_{i=2}^{v}\frac{\sin^2(\pi/v)}{\sin\left[(i-1)(\pi/v)\right]} \qquad M = 2\sin(\pi/v)$$

$$\dot{x}^2 + \dot{y}^2 + \dot{z}^2 = 2U(x,y,z) - C$$

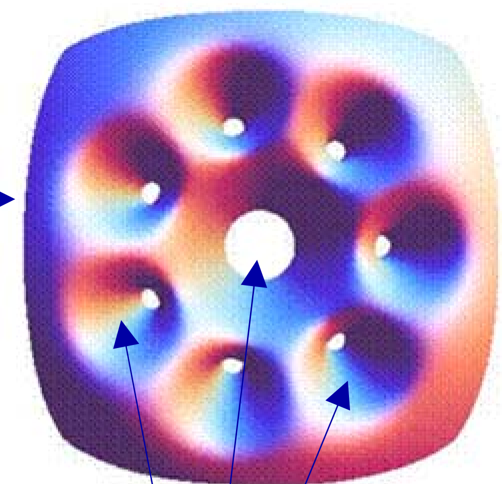
Jacobian-type integral of motion

ZERO - VELOCITY CURVES AND SURFACES IN THE PLANAR MOTION FOR THE GRAVITATIONAL CASE (e = 0)

- **Z.V.Curves:** $2U(x,y)=C$
- **Considering a third axis which counts the values of C, we obtain, for each zero-velocity diagram, a corresponding three-dimensional plot which is called zero-velocity surface of the particle planar motion .**



Zero-velocity surface

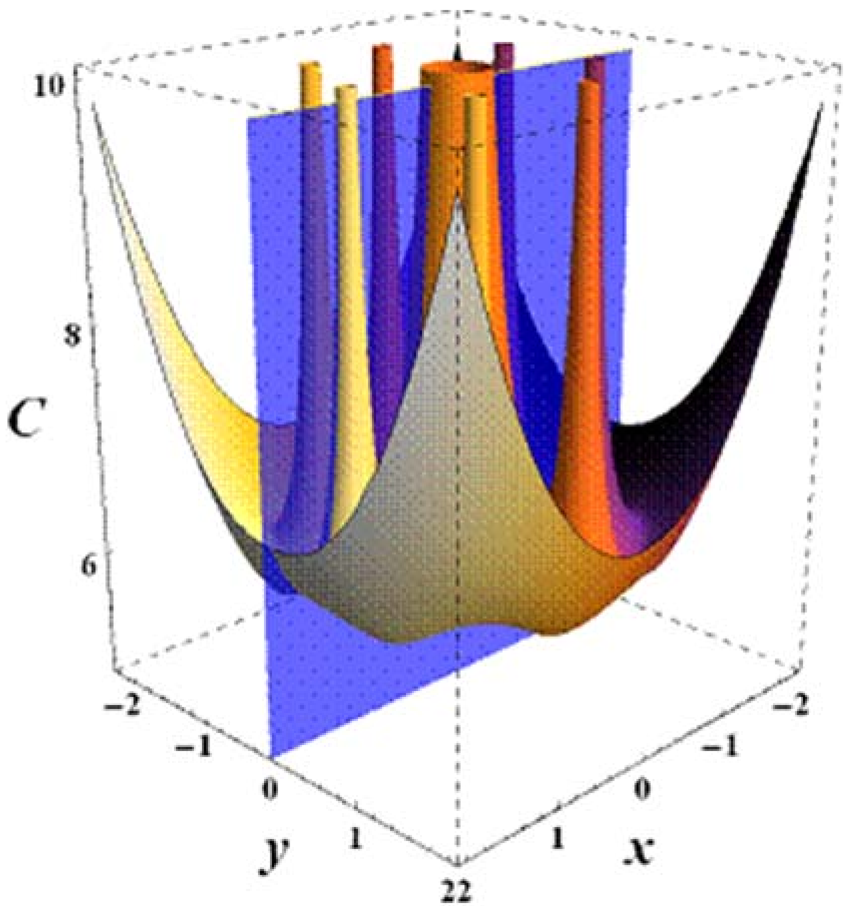


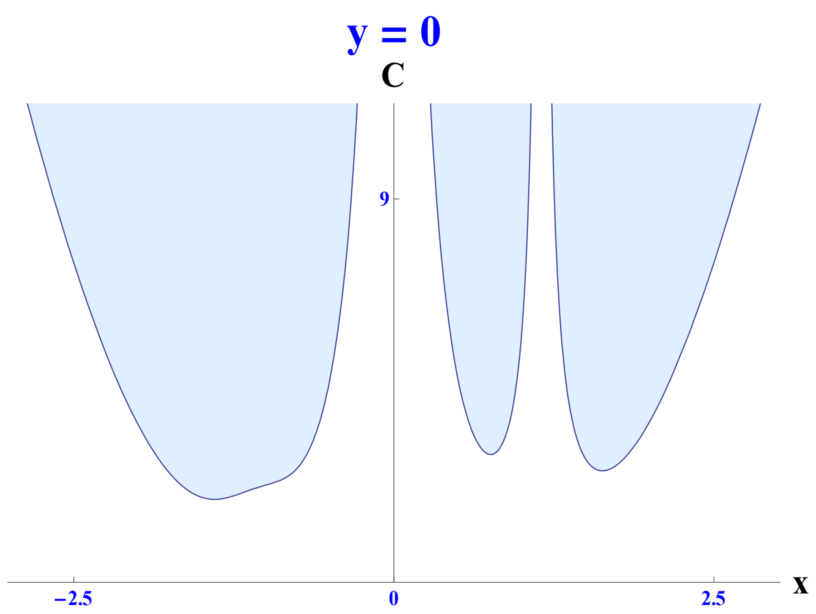
Zero-velocity curves

Areas of permitted motion

The diagrams (x, C) when e=0

- They are obtained by intersecting surface $C = C(x, y)$ with plane y=0
- They reveal the equilibria
- They limit the regions of the phase space of initial conditions where planar solutions exist.

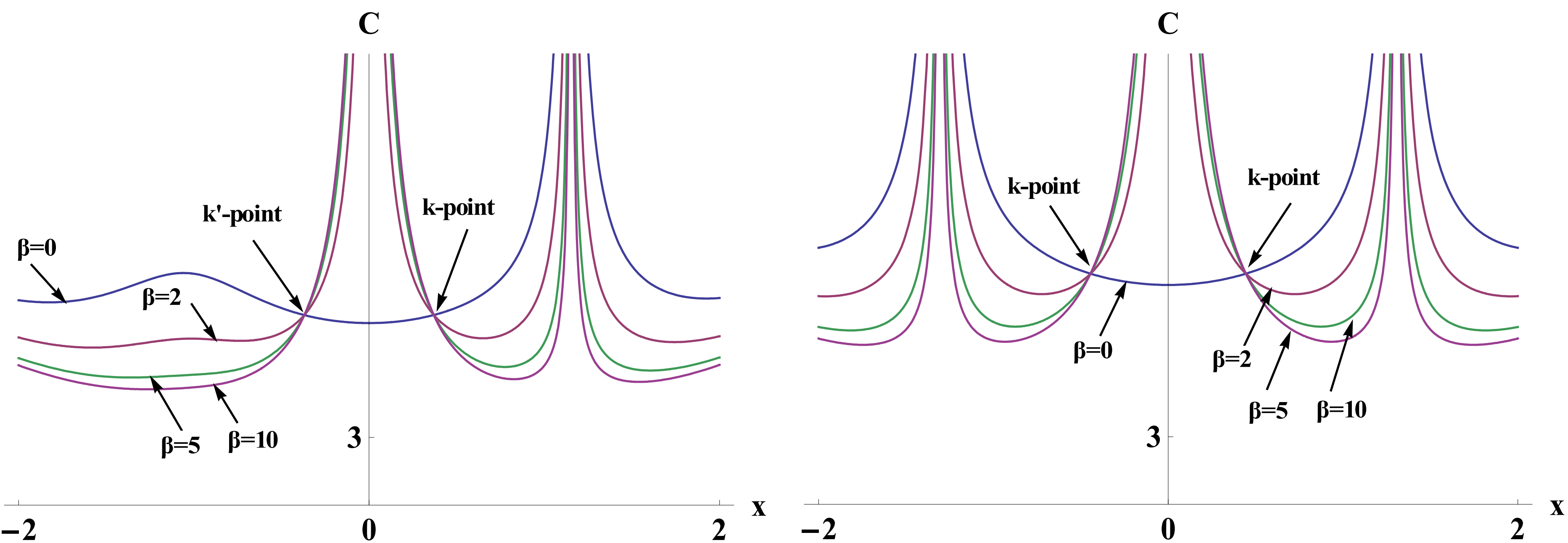




The colored regions, are the regions of the non-permitted motion

Focal points in the gravitational case

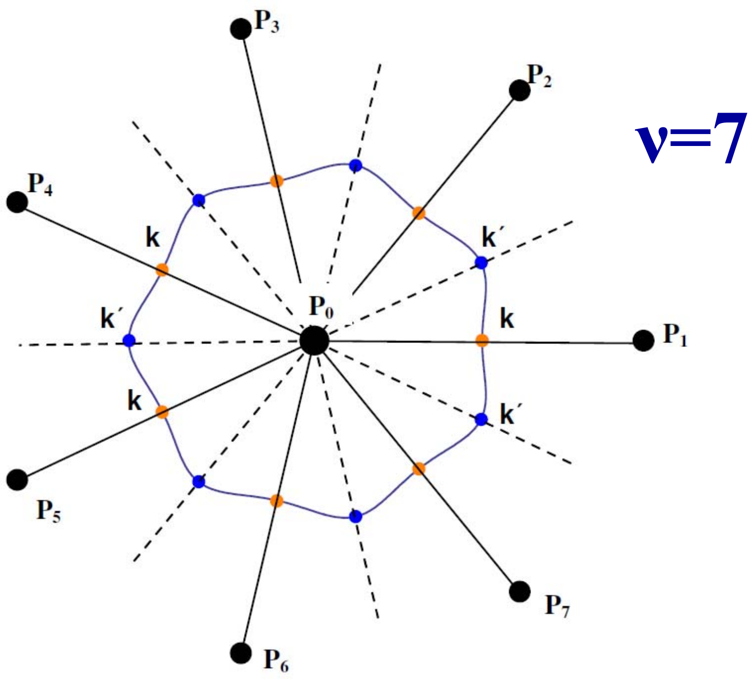
- The zero-velocity curves of the x-C diagrams for a given ν and y=0, intersect in two points (k, k') which are called focal points (Kalvouridis 2004).
- If ν is odd all the types of equilibria appear while if ν is even only the collinear equilibria appear.
- The coordinates of these points do not depend on the value of β.

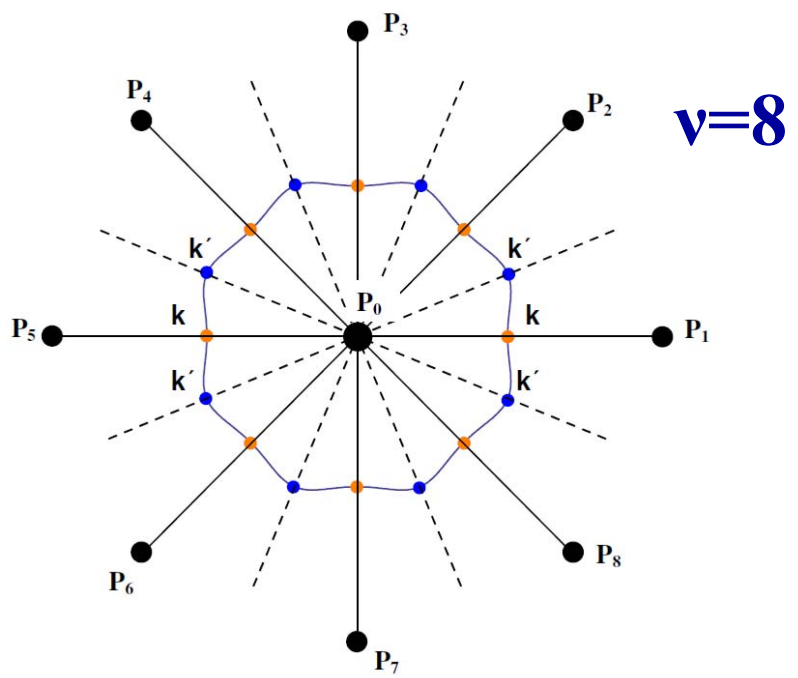


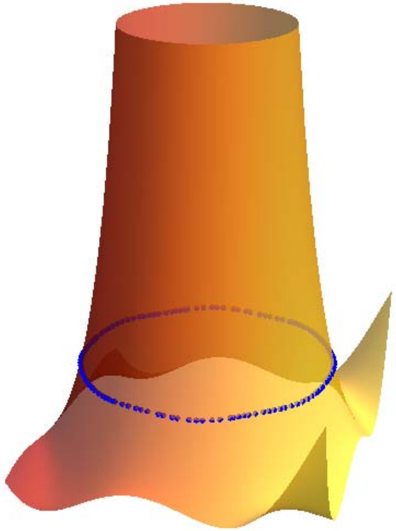
ν=7 , e=0 , various values of β   ν=8 , e=0 , various values of β

Collinear and triangular focal points

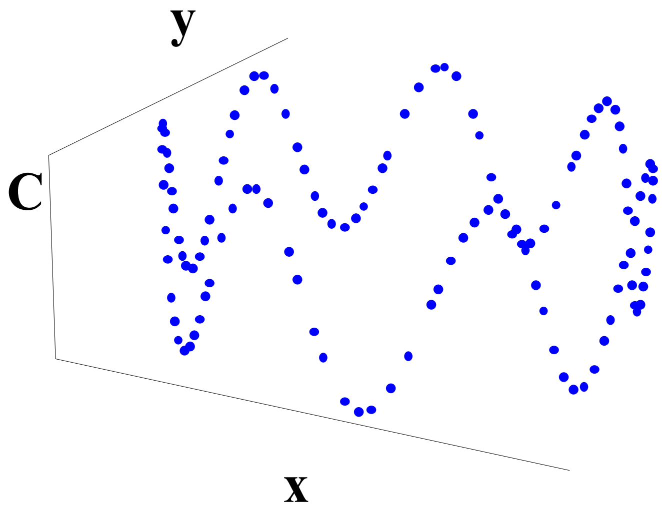
- We have observed that along the directions of the bisectors of the angles formed by the central primary and two successive peripheral primaries, lie the k'-type focal points (triangular focal points), while along the directions of the radii which connect the central primary to a peripheral one lie the k-points (collinear focal points).
- When ν is odd, two focal points, one collinear (k) and one triangular (k') with different coordinates appear on the (x,C) diagram.
- When ν is even, then, because we use the line which connects the central to a peripheral primary as the x-axis, both focal points are collinear (k-points).



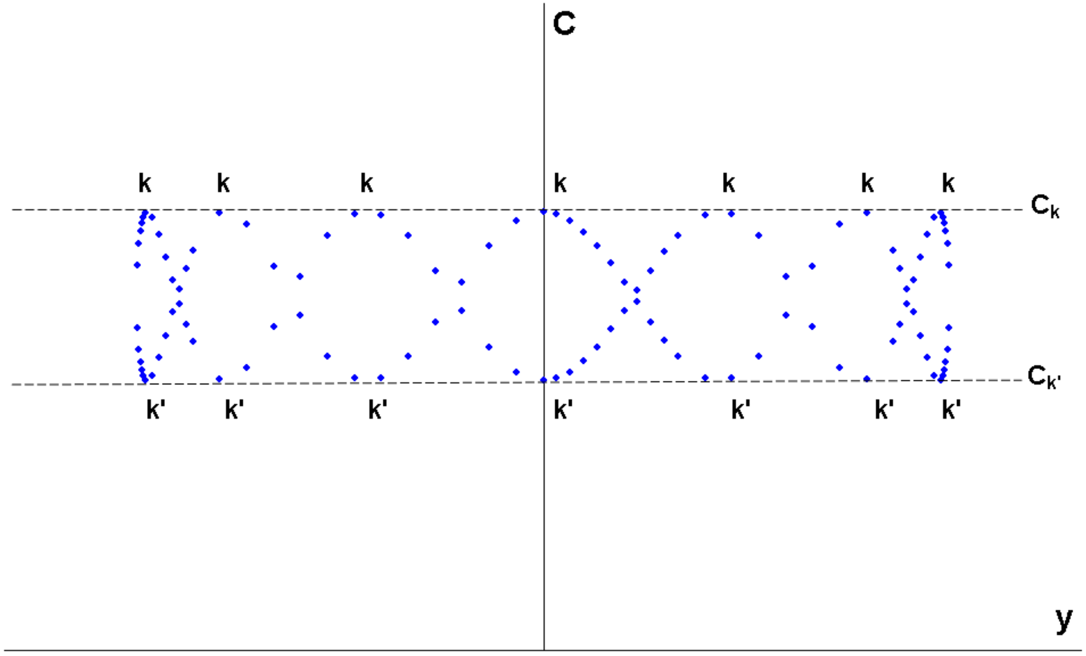




The focal curve of the zero-velocity surface C=C(x,y), around the central primary



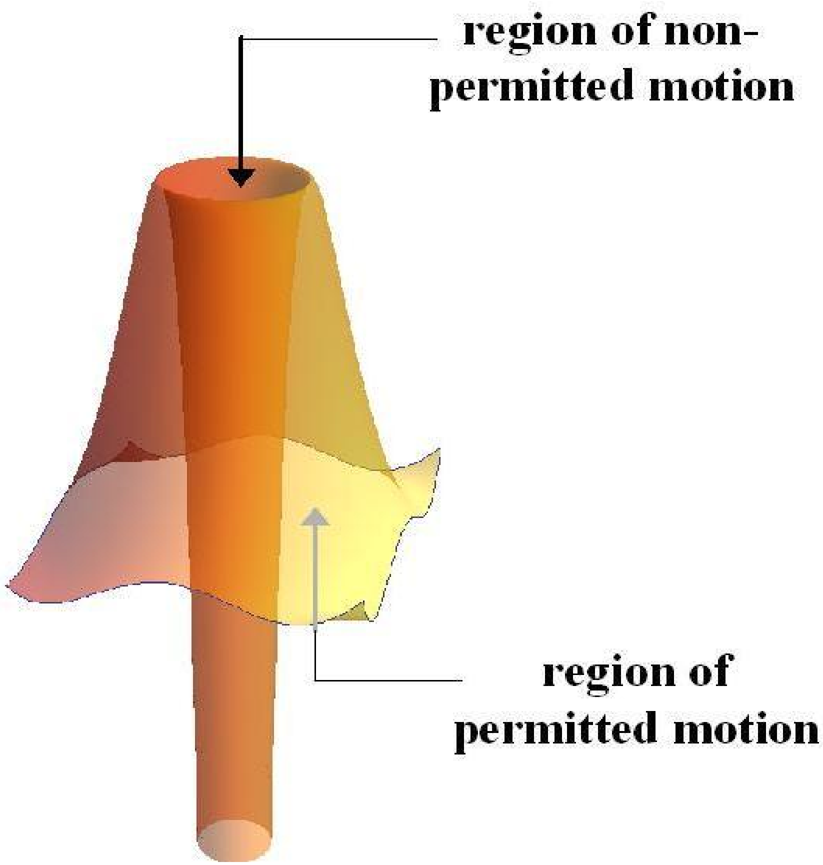
Wavy form of the focal curve in the (x,y,C) space

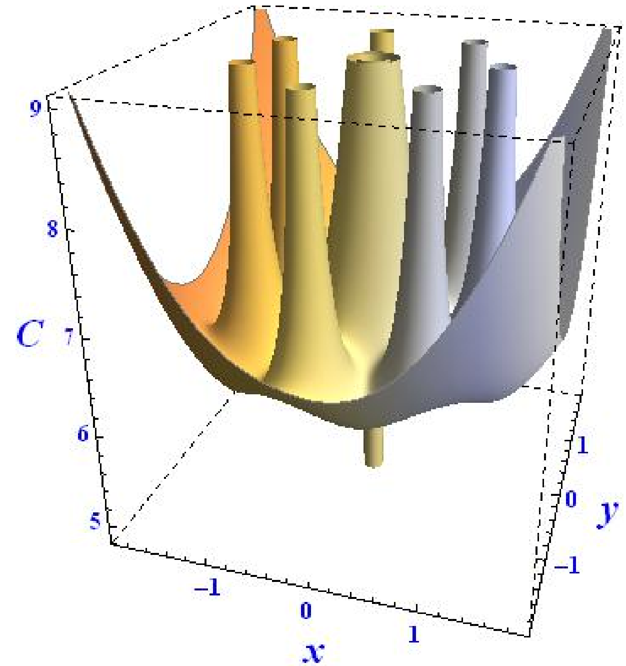


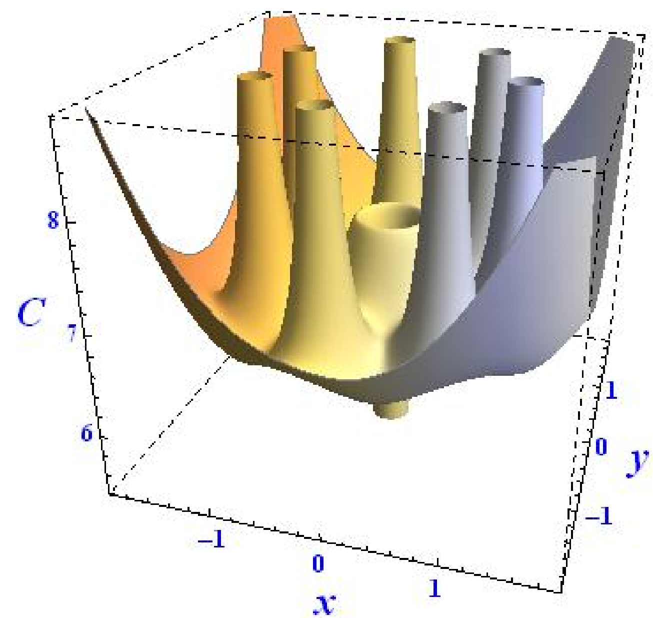
Projection of the focal curve on the Cy-plane

The zero-velocity surfaces in the case where e<0

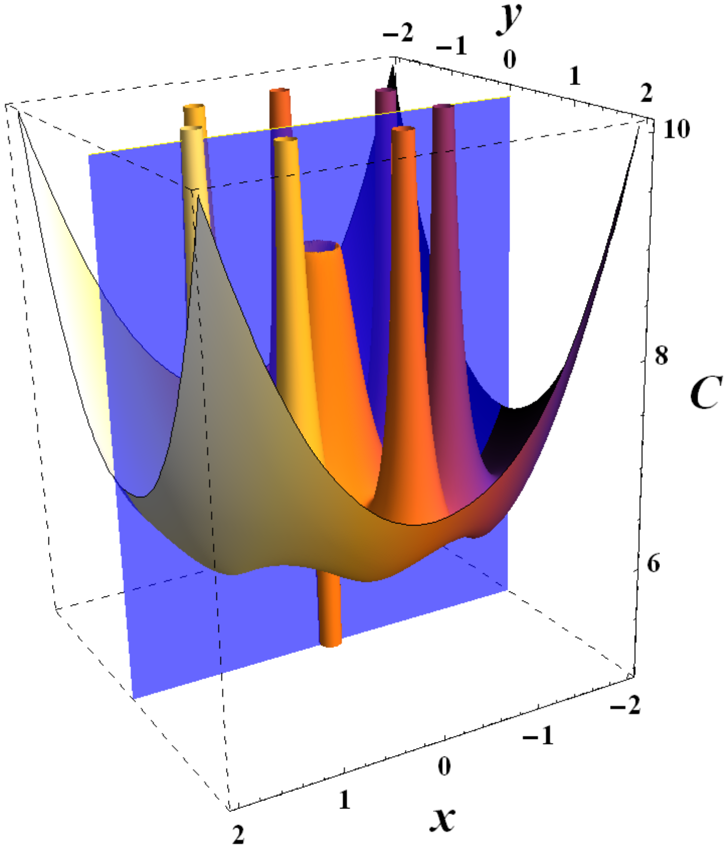
A "folding" of the central "chimney" that surrounds the central primary, is formed

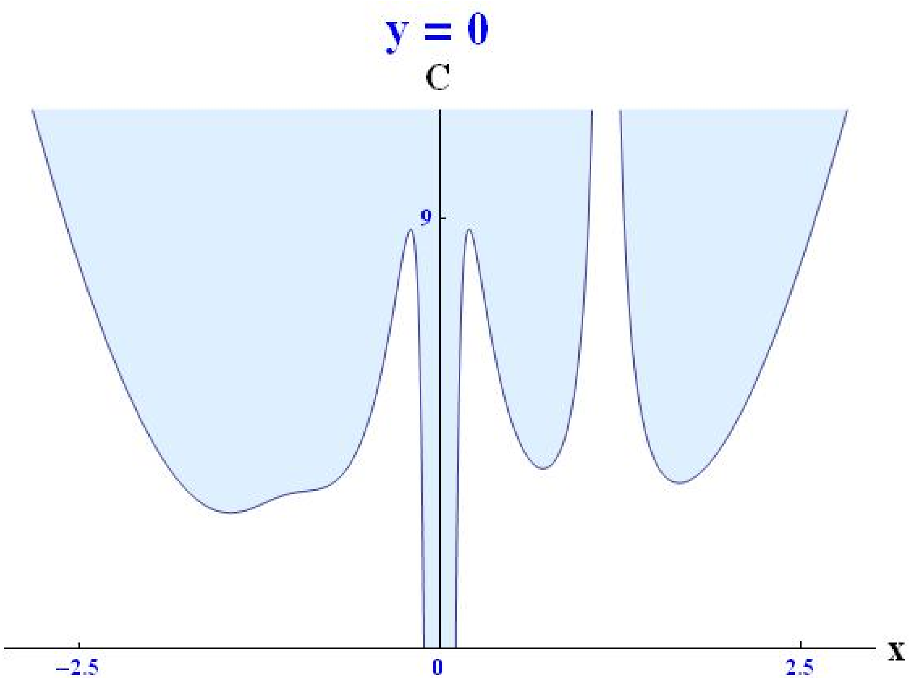






The diagrams (x, C) when e<0





The colored regions, are the regions of non-permitted motion

Focal points and focal curve when parameter β remains constant and parameter e varies. The function F_β

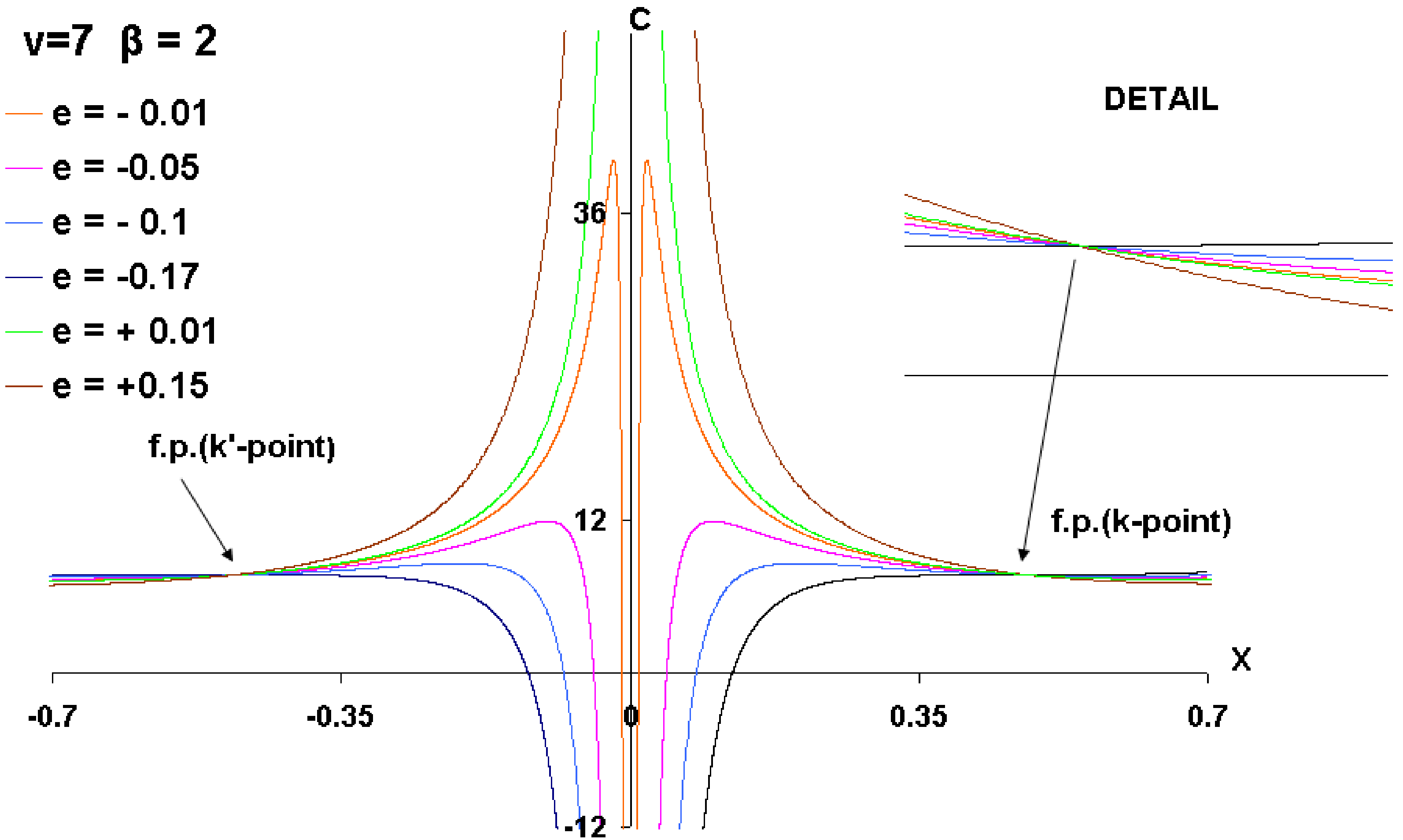
We assume that the parts of two zvc which evolve in the neighborhood of the central primary and are drawn by means of the Jacobian-type integral of motion for a given mass parameter β and two different values of parameter e, intersect. Then, and provided that $r_0 \neq 0$, the intersection points (focal points) of the two zvc will have the same coordinates x,y and C. Based on this property, we have found that the focal points satisfy the relation,

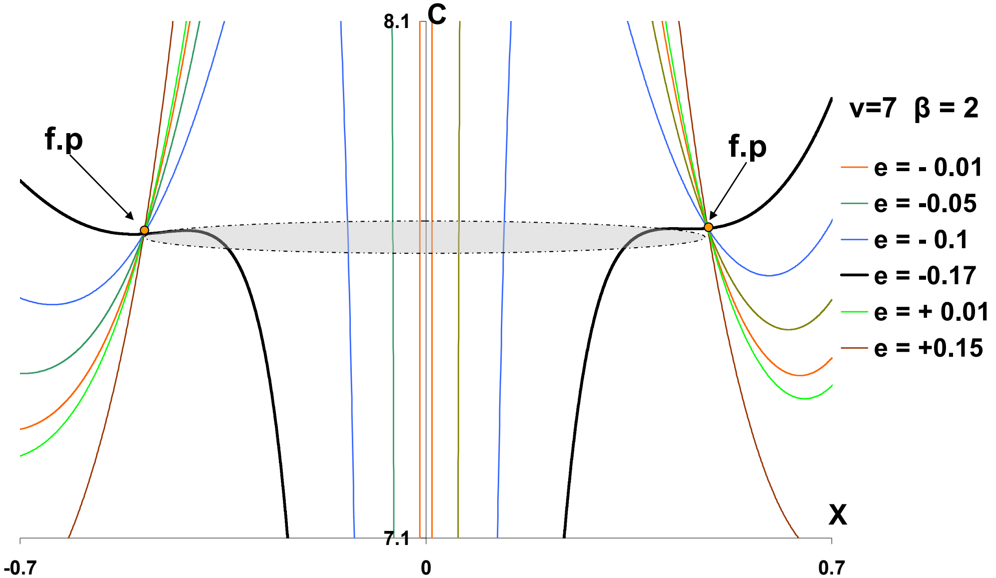
$$F_\beta(x; y_i, \beta) = \frac{r_0^2}{K - r_0\beta} \sum_{i=1}^{\nu} \frac{1}{r_i} - 1 = 0$$

$$K = \frac{\Lambda + \beta M^2}{2M^3}$$

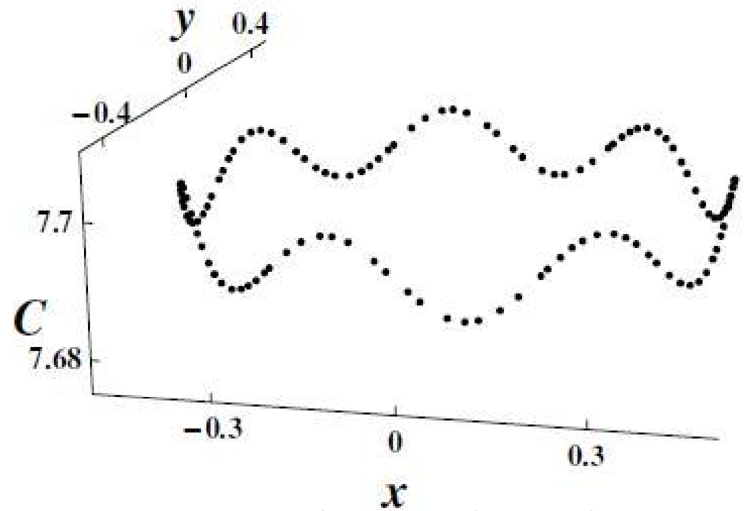
- The relation above does not depend on e.
- The focal points are roots of the above relation and belong to a continuous 3D curve (in the (x,y,C) space), the focal curve.

Superposition of the parts of the (x,C) diagrams near the central primary for a given value of β and various values of e

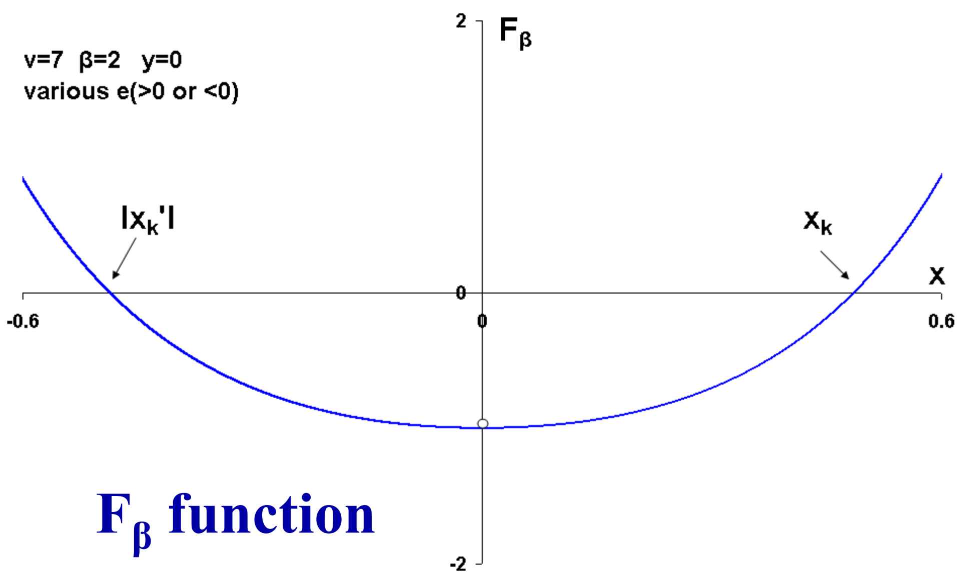




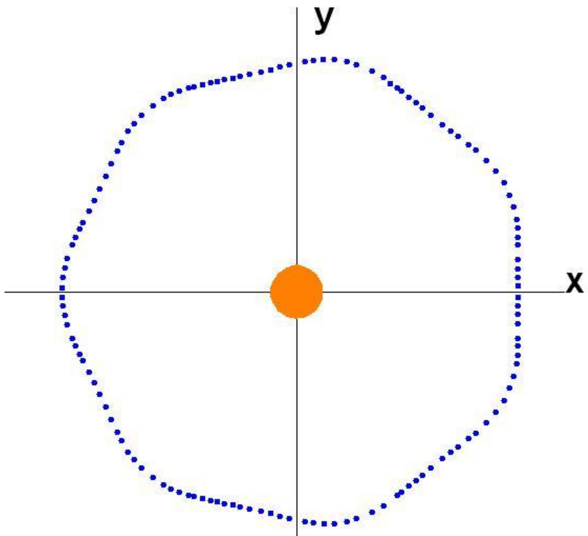
A draft sketch of the focal curve



Wavy form of the focal curve in the (x,y,C) space



F_β function



Projection of the focal curve on the xy-plane

Focal points and focal curves when parameter e remains constant and parameter β varies. The function F_e

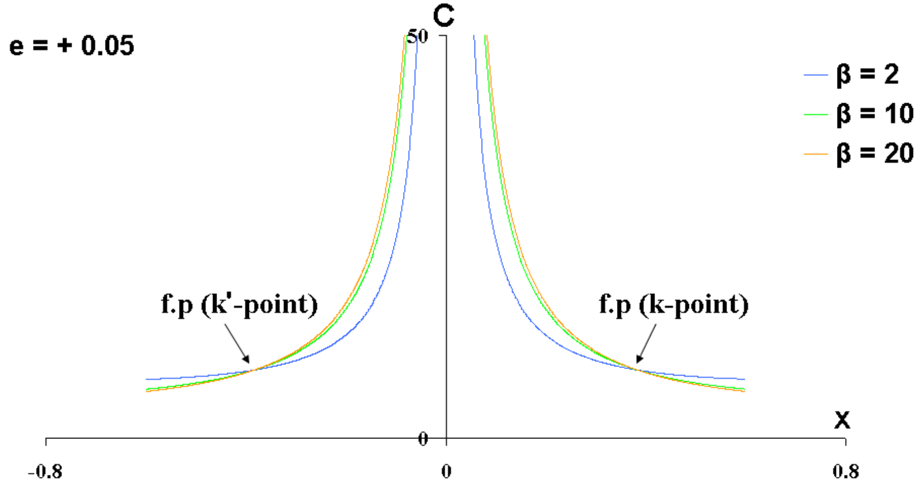
As in the previous case, we assume that the parts of two zvc which evolve in the neighborhood the central primary and are drawn by means of the Jacobian type integral of motion for a given Manev parameter e and two different values of parameter β, intersect. Then, and provided that $r_0 \neq 0$, the intersection points (focal points) of the two zvc will have the same coordinates x,y and C. Based on this property, we have found that in this case, the focal points satisfy the relation,

$$F_e(x;y_i,e) = \frac{M^2(1+2eM)}{\Lambda}\left(\frac{r_0^2}{r_0+e}\right)\sum_{i=1}^{\nu}\frac{1}{r_i} - 1 = 0$$

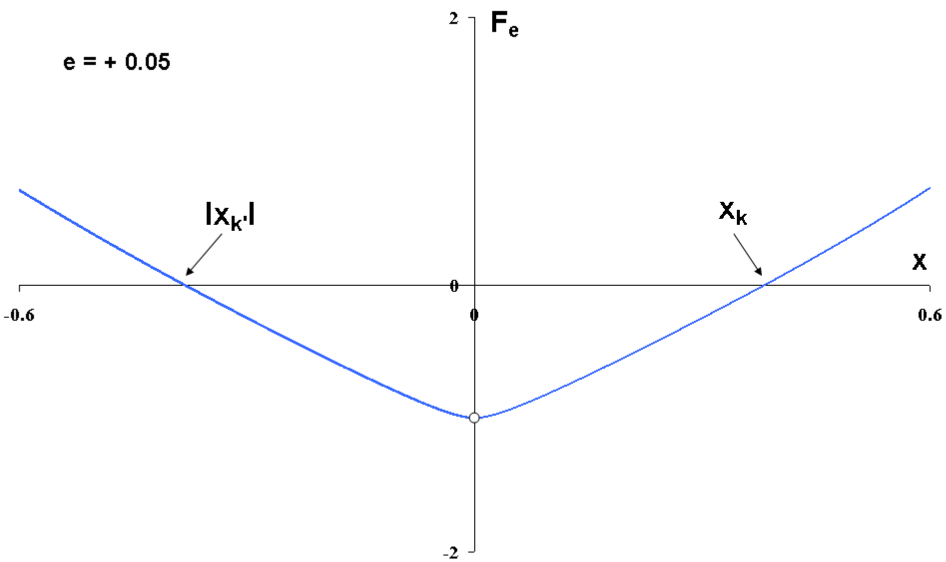
- The relation above does not depend on β.
- The focal points are roots of the above relation and belong to a continuous 3D curve (in the (x,y,C) space), the focal curve.

Focal points for a given value of e and the corresponding function $F_e(x)$

(I) Case where e>0



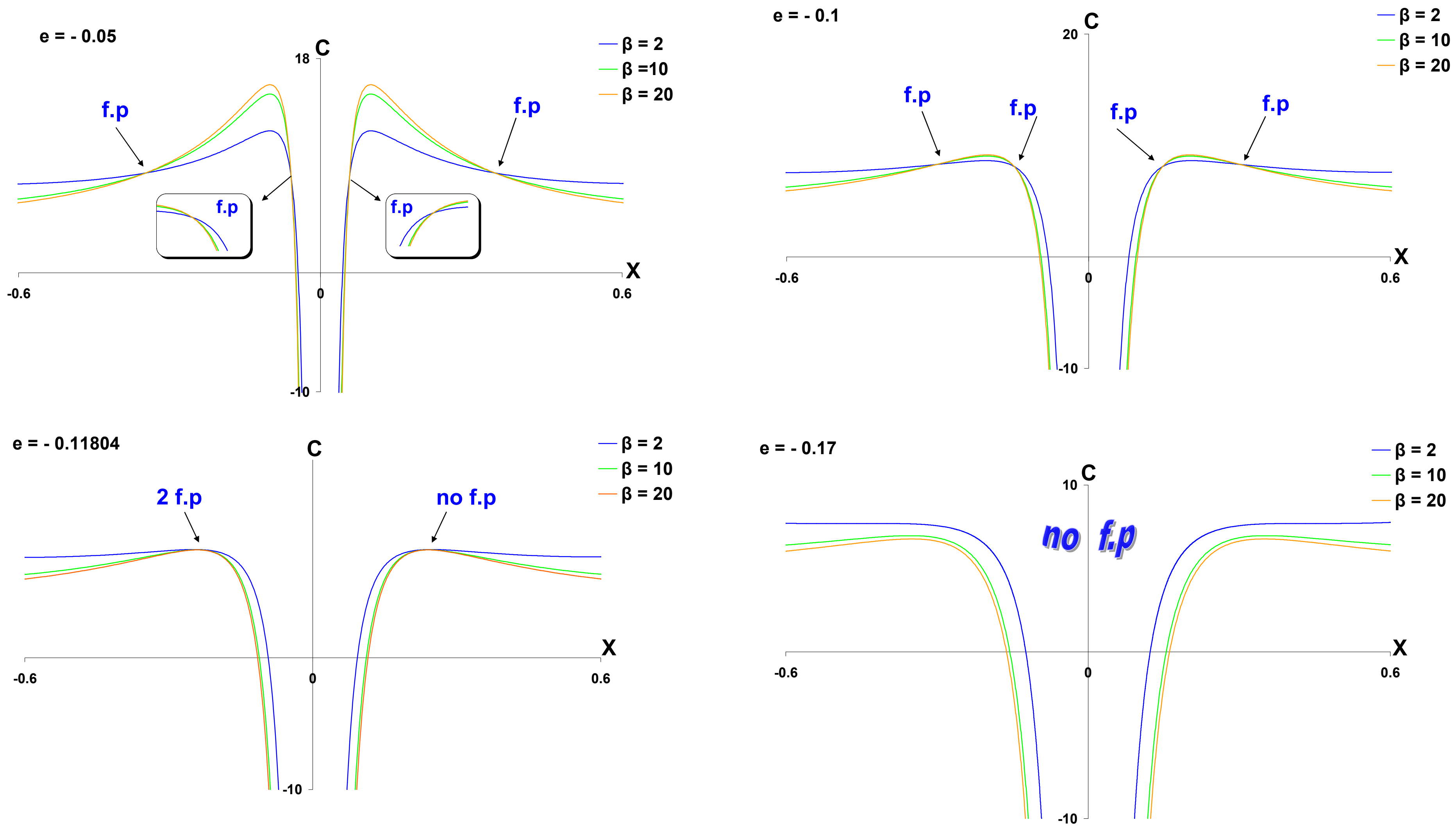
Zero-velocity curves (x-C) and focal points for ν=7, e=0.05 , y=0 and various values of mass parameter β.

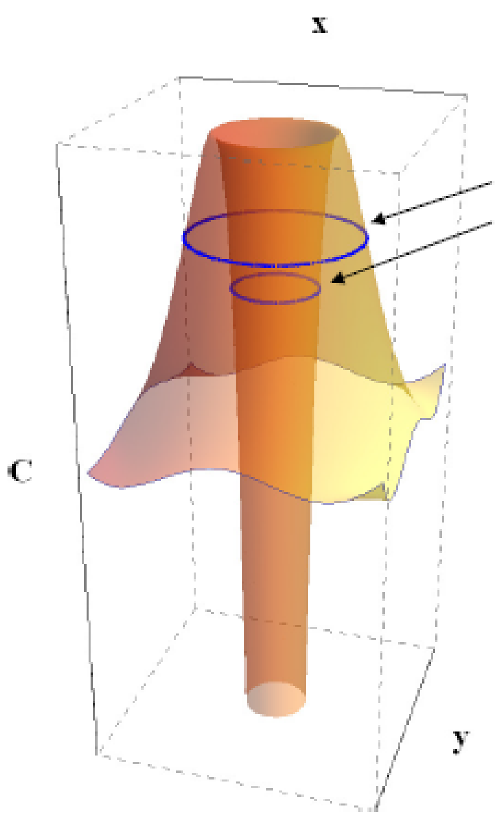


The function $F_e(x)$ for various values of β

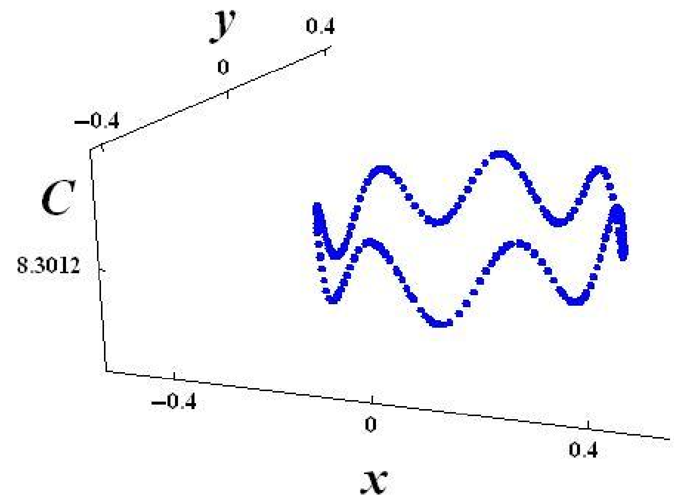
(II) Case where e<0

Evolution of the focal points with parameter e for ν=7 and y=0

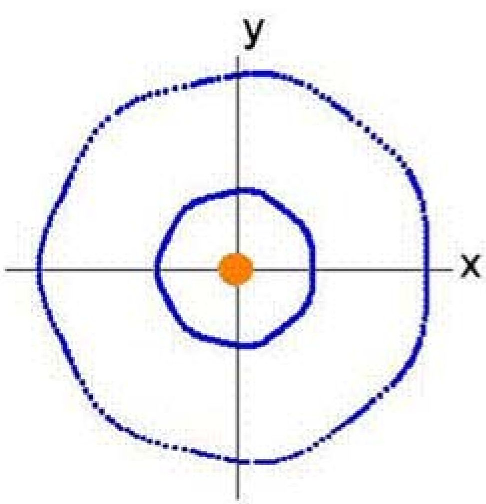




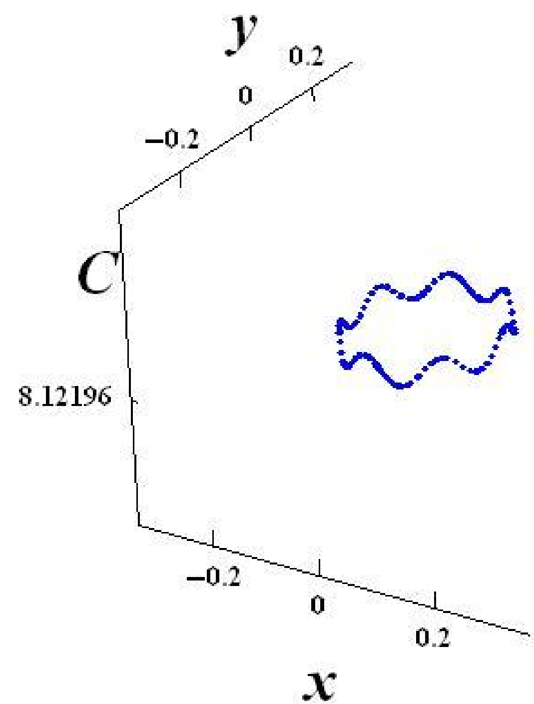
The zero-velocity surface near the central primary when e<0 and the two focal curves



Detail of the external focal curve



Projection of the focal curve on the xy-plane



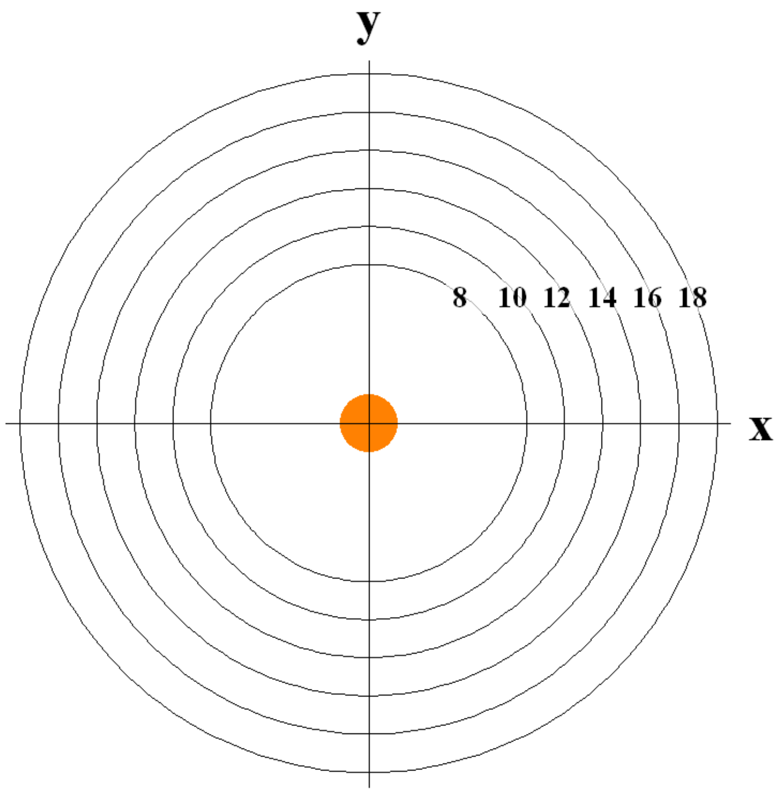
Detail of the internal focal curve

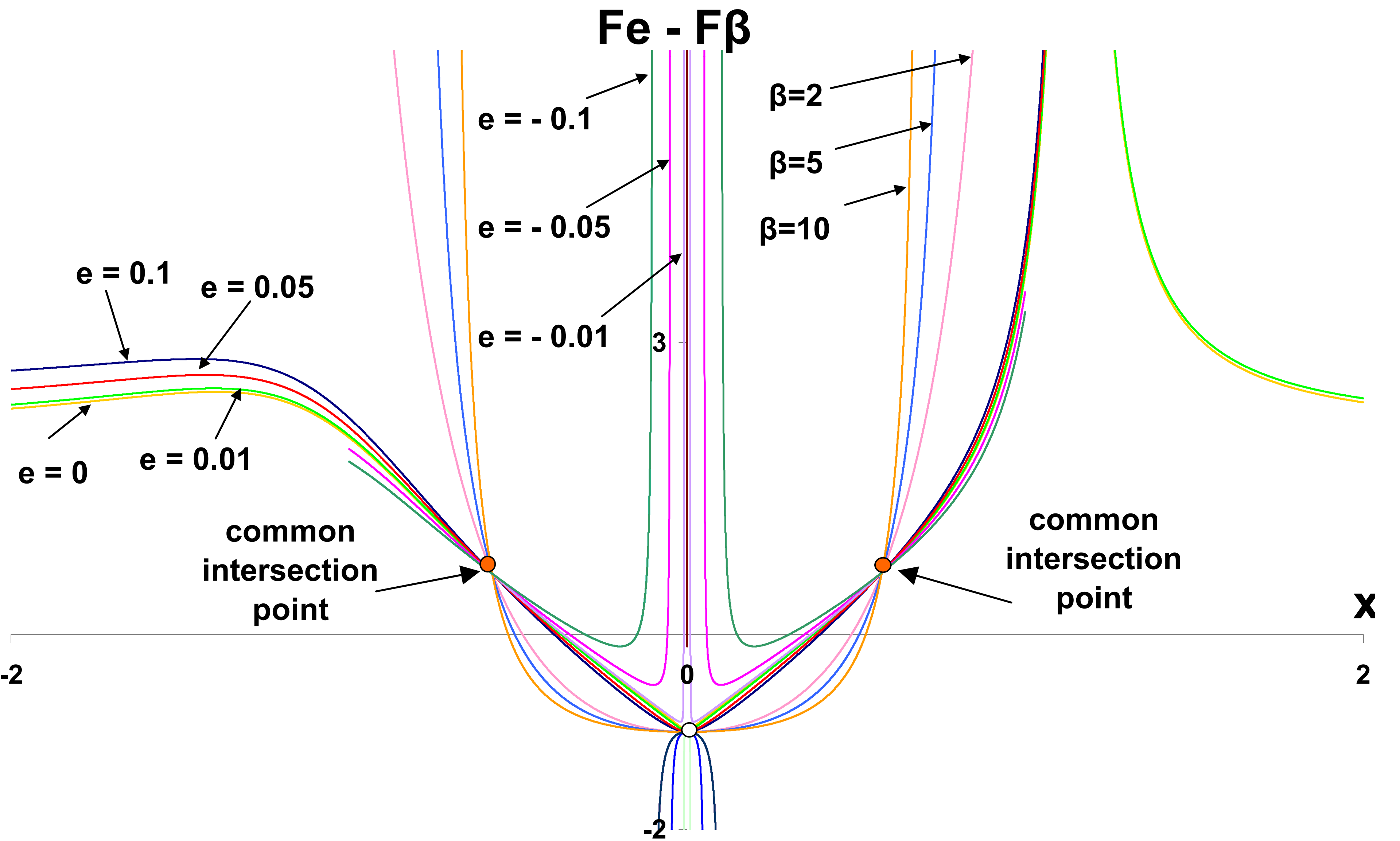
Common intersection points of functions F_β and F_e

We also found that functions F_β and F_e have common intersection points in the plane x-F which for a given ν satisfy the relation ,

$$r_0 = \frac{1}{2M}$$

- Which is the equation of a circle.
- These points do not depend on either parameters β or e and only depend on the number ν of the peripheral primaries.





Fe - Fβ
e = - 0.1
e = - 0.05
e = - 0.01
β=2
β=5
β=10
e = 0.1
e = 0.05
e = 0
e = 0.01
3
common intersection point
common intersection point
x
-2
0
2
-2

Further investigation of the focal points and curves in improved ring-type models

- Recently, we have investigated a more general version of the above model where besides the central primary with Manev parameter e, all the peripheral primaries create a Manev potential with a new parameter e' (positive or negative) common for all the peripheral primaries.
- We have found that in this case, the focal points and curves appear not only around the central primary, but also in the parts of the zero-velocity surface C=C(x,y) which evolves around each peripheral primary.

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THANK YOU VERY MUCH FOR YOUR ATTENTION