A Newtonian problem as a guide for relativistic astrophysics

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Overview

1. The Euler's problem
2. Similarity to Kerr
3. Exploiting this similarity

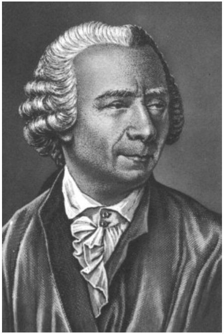
Euler $\sim$ 1760

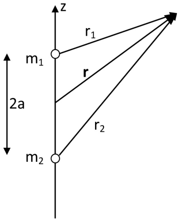
- A very simple Newtonian problem with interesting physical properties: The gravitational field of two fixed point masses.

$$V(r_1, r_2) = -\frac{Gm_1}{r_1} - \frac{Gm_2}{r_2} = -\frac{Gm_1}{|\mathbf{r} - a\hat{\mathbf{z}}|} - \frac{Gm_2}{|\mathbf{r} + a\hat{\mathbf{z}}|}$$

- The problem is better described in spheroidal coordinates:

$$\xi = \frac{r_1 + r_2}{2a}\ , \ \eta = \frac{r_2 - r_1}{2a}$$





Property similar to Kerr #1

- Orbits of test bodies in Euler's potential are described by an integrable problem, characterized by 3 integrals of motion:

$$E(\text{total energy}),$$
$$L_z(\text{z-angular momentum}),$$
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  ... just like in Kerr.
- Integrability condition renders the equations of motion seperable through the Hamilton-Jacobi technique (as in Kerr).
- The orbits revolve around the axis of symmetry, they oscillate radially, and with respect to the equatorial plane, but the equatorial plane is repulsive! (the OPPOSITE of Kerr).

Property similar to Kerr #2

Hold on!

There is a basic difference, behind this different behavior: The Euler's problem (the potential) is prolate, while Kerr is oblate. The quadrupole moment of Kerr is $M_2^{(K)} = -ma^2$, while the one of Euler is $M_2^{(E)} = +ma^2$.

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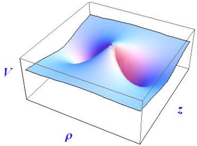
Here is the clue to make them look more similar: $a \to ia$.
This is not physically realizable, but it has a perfectly real potential (at least when the masses are equal $m_1 = m_2 = M/2$):

$$V(\mathbf{r}) = -\frac{GM}{\sqrt{2}R^2}\sqrt{R^2 + r^2 - a^2} = -\frac{GM\xi}{a(\xi^2 + \eta^2)}$$

with $R = \sqrt[4]{(r^2 - a^2)^2 + (2a\mathbf{r}\cdot\hat{\mathbf{z}})^2}$.



Now it is an oblate potential with attractive equatorial plane (as Kerr).

Property similar to Kerr #3

The 3rd integral of motion in Kerr (the Carter constant)

For $\mu_{\text{test body}} = 1$

$$Q^{(K)} = u_\theta^2 + \cos^2\theta \left[a^2(1 - E^2) + L_z^2/\sin^2\theta\right]$$

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The Euler's 3rd integral of motion

If we use the following physical transformations

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Q yields the following form for a test body with unitary mass

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What a surprise!

Property similar to Kerr # 3...

... continued

$Q^{(E)}$ could also be written in terms of ξ and p_ξ, as well. By replacing $a\xi \to r$ and omit higher order terms of M/r, it coincides again with the corrsponding expression of $Q^{(K)}$.

Property similar to Kerr # 3...

... continued

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Now the identification is not complete - it's approximate.

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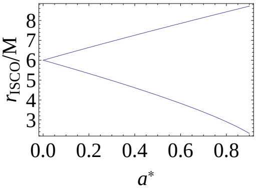
- For Kerr: $$\frac{r_{ISCO}}{M} = 3 + Z_2 \mp \sqrt{(3 - Z_1)(3 + Z_1 + 2Z_2)}$$

  with $Z_1 = 1 + \sqrt[3]{1 - a^{\star\,2}}\left(\sqrt[3]{1 - a^{\star}} + \sqrt[3]{1 + a^{\star}}\right)$, $Z_2 = \sqrt{3a^{\star\,2} + Z_1^2}$, and $a^{\star} = a/M$.



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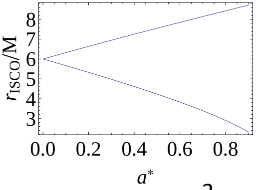

- For Euler: $$\frac{r_{ISCO}}{M} = a\xi_{ISCO} = \sqrt{3}\frac{a}{M} = \sqrt{3}a^\star$$

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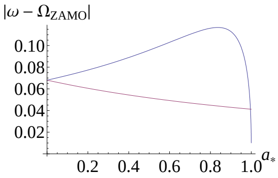

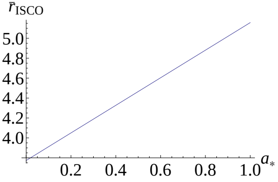

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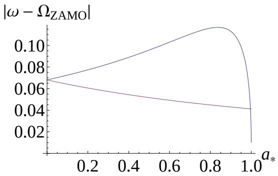

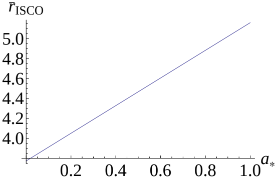

The final plot of equivalent r_{ISCO} for Kerr is almost linear as in Euler.
(The initial value and the slope is different though.)

Property similar to Kerr # 5

Since both problems are separable, one could obtain formal expressions for the fundamental frequencies of them.

The frequencies

$$\Omega_r = \Omega_\xi = \frac{\pi K(k)}{K(k)Y + a^2 z_+ X\left[K(k) - E(k)\right]}$$

$$\Omega_\theta = \Omega_\eta = \frac{(\pi/2)\beta\sqrt{z_+}X}{K(k)Y + a^2 z_+ X\left[K(k) - E(k)\right]}$$

$$\Omega_\phi = \frac{\left(K(k)Z + L_z\left[\Pi(\pi/2), z_-, k) - K(k)\right]X\right.}{K(k)Y + a^2 z_+ X\left[K(k) - E(k)\right]}$$

with $\{X, Y, Z\}$ radial integrals related to the r-potential V_r, $\{K, E, \Pi\}$ elliptic integrals related to the azimuthal-potential V_θ, $\beta^2 = a^2(1 - E^2) = 2a^2 E_N$ and $k = \sqrt{z_-/z_+}$ ($z_\pm$ are the roots of V_θ).

Properties similar to Kerr # 5...

The expressions are the same (at least formally) although some radial integrals Z are somewhat different elliptic integrals, e.g.

$$Z^{(K)} = \int_{r_1}^{r_2} \frac{\left[L_z r^2 - 2Mr(L_z - aE)\right]}{(r^2 + a^2 - 2Mr)\sqrt{V_r(r)}} dr$$

while

$$Z^{(E)} = \int_{a\xi_1}^{a\xi_2} \frac{L_z r^2}{(r^2 + a^2)\sqrt{V_\xi(r)}} dr$$

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Note: The frequencies in Kerr are directly observable through GWs.

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- Following their arguments we have shown that Euler's problem pocesses exactly the same property. We have shown theoretically the existence of identical frequencies but we are still searching for a specific example.
- Due to axisymmetry of orbits in Euler, things are more easy to analyze there. This is an example of a property of Kerr that could have been discovered by analyzing the Euler problem, first.

Properties similar to Kerr # 7

Recently, (Glampedakis, Apostolatos CQG30, 055006, 2013) another common property of Euler's potential and Kerr background was revealed:

- The wave equation in Euler's background

  $$\Box\Psi = -\kappa V \Psi$$

  have a striking similarity to the perturbation equations for a scalar field in Kerr.

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$$\Box\Psi = -\kappa V \Psi$$

  have a striking similarity to the perturbation equations for a scalar field in Kerr.

- The wave equation is separable in both cases. The azimuthal parts are identical, while the radial one is similar but not identical. Furthermore the difference shows up in the potential of the wave equation beyond the second to higher order term.

Properties similar to Kerr # 8,...

There are a few other similarities that were found much earlier (1970s).

8. The two fields are characterized by the same set of multipole moments (Geroch 1970)

$$M_{2l} = M(-a^2)^l, \; \left[S_{2l+1} = Ma(-a^2)^l \text{ (for Kerr)}\right]$$

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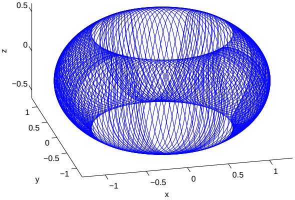
10 Keres (1967) and Israel (1970) showed that the Newtonian mass distribution resembling Kerr's effective source density is that of oblate Euler's problem.

Properties similar to Kerr # 8,...

11 A Kerr geodesic orbit could be "circular" (meaning spherical $r = \mathrm{const}$). Exactly the same kind of orbits $\xi = \mathrm{const}$ (thus ellipsoidal in real space) exist in Euler's problem as well.



? ... probably more ...?

Are these similarities useful?

Fascinating, but
... are they of any use?

A toy-model for Kerr

Assuming we have an analytic toy-model for Kerr, we could use it to investigate properties of a dynamic Kerr background.

1. Circular goes to circular?

Kenefick and Ori (PRD1996) have presented a quite obsure argument according to which an initially "circular" orbit in Kerr will drift adiabatically to "circular" orbits under radiation reaction. Euler's problem is a nice testbed to check both analytically and numerically their argument. (The analytical part has already been confirmed.)

Relying on Euler for prognostics

Radiation reaction force for Kerr is still unknown (a huge analytical effort has been made during the late decade to compute it approximately). There are still open issues on modelling the radiating wave signals, with sufficient accuracy, to detect and monitor them.

2. Orbits close to resonances?

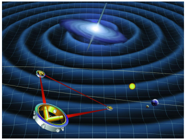
- Flanagan and Hinderer (PRD2008, PRD2012) have shown that our naive evloution of orbits in Kerr near resonances could be disastrous for our ability to even detect them (especially in EMRIs).
- Euler ammended by a dissipative (radiation-like) artificial force could give us at least a qualitative answer. We have suggested such a force

$$\mathbf{F}_{RR} = -\epsilon F_G u^4 \mathbf{u}$$

(... under investigation)

Relying on Euler for prognostics...

Could one use GW signals to check if relativistic astrophysics/GR is right?



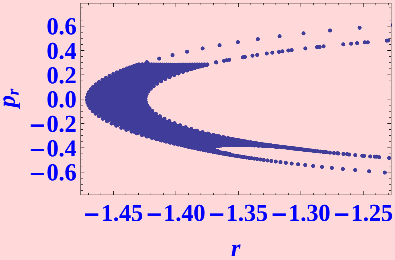
3. Non Kerr objects - non Euler objects

- Apostolatos, Gerakopoulos, Contopoulos (PRL2009, PRD2010) suggested applying the KAM theorem and Birkhoff chain of islands (characteristic behavior of slightly non-integrable systems) to check for non-Kerr astrophysical objects through GWs.
- A suitably perturbed Euler (with a small mass at the origin) could be the analogue of a perturbed Kerr. Some artificial self-force could adiabatically make an orbit pass through a resonance and its consequences could be studied.

Relying on Euler for prognostics...

3. Non Kerr objects - non Euler objects...

- Preliminar studies show that the passage through a resonance is quite enhanced, compared to what we initially thought (and used in our paper). The plateau effect in the evolution of the ratio of the observed frequencies might be much more pronounced.



Thank you
... for your attention

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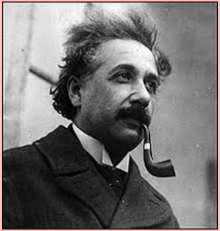

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