Chern-Simons Modified Gravity

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- Then talk about vacuum solutions of the Chern-Simons theory in both the non-dynamical and the dynamical formalism

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- Then talk about vacuum solutions of the Chern-Simons theory in both the non-dynamical and the dynamical formalism
- We will focus **only** on axisymmetric (static and stationary) solutions that are also solutions of General Relativity

Mathematical Formalism

$$S \equiv S_{EH} + S_{CS} + S_{\theta} + S_{\text{mat}} \tag{1}$$

With :

$$S_{EH} = \kappa \int_{\mathcal{V}} \sqrt{-g} R d^4x \tag{2a}$$

$$S_{CS} = \alpha \frac{1}{4} \int_{\mathcal{V}} \sqrt{-g} \theta \, {}^*R\, R d^4x \tag{2b}$$

$$S_{\theta} = -\beta \frac{1}{2} \int_{\mathcal{V}} \sqrt{-g} \left[g^{ab} \left(\nabla_a \theta\right) \left(\nabla_b \theta\right) + 2V(\theta) \right] d^4x \tag{2c}$$

$$S_{\text{mat}} = \int_{\mathcal{V}} \sqrt{-g} \mathcal{L}_{\text{mat}} d^4x \tag{2d}$$

Where :

$$ {}^*R\,R \equiv {}^*R^{a\ cd}_{\ b} R^b{}_{acd} $$

is a quantity called *Pontryagin Density*.
It is of topological nature and connected to the following quantity, which is called the *Topological Current*, :

$$ K^a \equiv \epsilon^{abcd} \Gamma^n{}_{bm} \left(\partial_c \Gamma^m{}_{dn} + \frac{2}{3} \Gamma^m{}_{cl} \Gamma^l{}_{dn} \right) $$

by :

$$ {}^*R\,R = 2 \nabla_a K^a $$

We ask for :

$$\delta S = 0 \tag{1}$$

which yields :

$$\begin{aligned} \delta S &= \kappa \int_{\mathcal{V}} \sqrt{-g} \left(R_{ab} - \frac{1}{2} g_{ab} R + \frac{\alpha}{\kappa} C_{ab} - \frac{1}{2\kappa} T_{ab} \right) \delta g^{ab} d^4x \\ &+ \int_{\mathcal{V}} \sqrt{-g} \left[\frac{\alpha}{4} {}^*R\,R + \beta \Box \theta - \beta \frac{dV}{d\theta} \right] \delta\theta d^4x \\ &+ \Sigma_{EH} + \Sigma_{CS} + \Sigma_\theta = 0 \end{aligned} \tag{2}$$

Σ_{EH}, Σ_{CS} and Σ_θ are surface terms which arise from integration by parts. They are generally vanishing.

So the field equations are :

$$G_{ab} + \frac{\alpha}{\kappa} C_{ab} = \frac{1}{2\kappa} T_{ab} \tag{1}$$

and

$$\beta \Box \theta = \beta \frac{dV}{d\theta} - \frac{\alpha}{4} {}^*R\, R \tag{2}$$

G_{ab} is just the Einstein tensor without a cosmological constant and C_{ab} is defined as follows :

$$C^{ab} \equiv u_{cd}\, {}^*R^{d(ab)c} + u_d \epsilon^{dce(a} \nabla_e R^{b)}{}_c \tag{3}$$

where

$$u_d \equiv \nabla_d \theta \ , \ \ u_{cd} \equiv \nabla_c \nabla_d \theta \tag{4}$$

The C-Tensor we defined, earlier in Eq. (3), by :

$$C^{ab} \equiv u_{cd}\,{}^{*}R^{d(ab)c} + u_d \epsilon^{dce(a} \nabla_e R^{b)}{}_c \tag{5}$$

is traceless and one can prove that :

$$\nabla_a C^{ab} = -\frac{1}{8} u^b\,{}^{*}R\,R \tag{6}$$

this fact, in vacuum, leads directly to the evolution of θ shown in Eq. (2).

In the above we used T_{ab} as the Energy-Stress tensor, this is actually :

$$T_{ab} = -\frac{2}{\sqrt{-g}}\left(\frac{\delta \mathcal{L}_{\text{mat}}}{\delta g^{ab}} + \frac{\delta \mathcal{L}_{\theta}}{\delta g^{ab}}\right) \tag{7}$$

This means that T_{ab} consists of two parts. One is :

$$T_{ab}^{\text{mat}} = -\frac{2}{\sqrt{-g}}\frac{\delta \mathcal{L}_{\text{mat}}}{\delta g^{ab}} \tag{8}$$

and the other is, explicitly, given by S_{θ} :

$$T_{ab}^{\theta} = \beta\left[\left(\nabla_a\theta\right)\left(\nabla_b\theta\right) - \frac{1}{2}g_{ab}\left(\nabla_a\theta\right)\left(\nabla^a\theta\right) - g_{ab}V\left(\theta\right)\right] \tag{9}$$

One can set :

$$\beta = 0 \tag{10}$$

This immediately yields :

$$T^{\theta}_{ab} \equiv 0 \tag{11}$$

and Eq. (2) is reduced to the consrtaint :

$$^{*}R\,R = 0 \tag{12}$$

We call the last one the *Pontryagin Contraint*. This contraint is, obviously, indepedent of θ.

Implications of the Non-Dynamical Formalism

One can connect the contraint with the Weyl tensor, through :

$$ {}^{*}R\,R = {}^{*}C\,C \tag{13} $$

$$ {}^{*}C\,C = \frac{1}{2}\epsilon^{cdef}C^{a}{}_{bef}C^{b}{}_{acd} \tag{14} $$

and then by :

$$ \left(C_{abcd} + \frac{i}{2}\epsilon_{abef}C^{ef}{}_{cd}\right)v^{b}v^{d} = E_{ac} + iB_{ac} \tag{15} $$

to :

$$ E_{ab}B^{ab} = 0 \tag{16} $$

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This means that we must have either :

- Purely electrical spacetimes ($B_{ab} = 0$)
- Purely magnetic spacetimes ($E_{ab} = 0$)
- Spacetimes where E and B are vertical

One can also see that the constraint can be written as :

$$\Im(\mathcal{J}) = \Im\left(\Psi_0\Psi_4 + 3\Psi_2^2 - 3\Psi_1\Psi_3\right) = 0 \tag{13}$$

for the Weyl spinor $\mathcal{J}$ and the NP scalars $(\Psi_0, \Psi_1, \Psi_2, \Psi_3, \Psi_4)$.
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This means that :

- Spacetimes of Petrov Type III, N and O satisfy the constraint
- Spacetimes of Petrov Type D, II, and I could violate it
- The Kerr spacetime and perturbations of it also violate the constraint, since $\Psi_1 = \Psi_3 = 0$ and $\Re\left(\Psi_2\right) \neq 0 \neq \Im\left(\Psi_2\right)$

If we consider β to be arbitrary, but non-zero, then θ must satisfy :

$$\beta \Box \theta = \beta \frac{dV}{d\theta} - \frac{\alpha}{4} \, {}^{*}R \, R \tag{14}$$

therefore, θ is a dynamic field, which contributes to the energy of the problem by :

$$T^{\theta}_{ab} = \beta \left[\left(\nabla_a \theta\right) \left(\nabla_b \theta\right) - \frac{1}{2} g_{ab} \left(\nabla_a \theta\right) \left(\nabla^a \theta\right) - g_{ab} V\left(\theta\right) \right] \tag{15}$$

The equations of motion of the Chern-Simons Modified Gravity are :

- Modified Einstein Equations

$$G_{ab} + \frac{\alpha}{\kappa} C_{ab} = \frac{1}{2\kappa} T_{ab} \tag{16}$$

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- Evolution Equation for the Coupling Field θ

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$$T^{\theta}_{ab} = \beta \left[(\nabla_a \theta)(\nabla_b \theta) - \frac{1}{2} g_{ab} (\nabla_a \theta)(\nabla^a \theta) - g_{ab} V(\theta) \right] \tag{18}$$

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Therefore in this context :

- θ is arbitrary and usually, chosen as to simplify the equations
- The constraint above radically reduces the space of solutions of the new theory, since it imposes ”heavy” geometrical restrictions

On the dynamical framework we suppose $\beta \neq 0$, therefore :

- θ is no longer arbitrary and its evolution is governed by

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- Since θ has its own equation of motion, it also introduces extra energy to the problem
- One should always solve that equation and then try to solve the modified Einstein equations
- Arbitrariness is not completely lifted as we introduce an arbitrary potential $V(\theta)$ - although this could be gauged away

Vacuum Solutions in the Non-Dynamical Framework

Consider the Schwarzschild line element :

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 \left(d\vartheta^2 + \sin^2\vartheta d\phi^2\right) \tag{17}$$

and also consider $\theta = \theta(t, r, \vartheta, \phi)$. Since ${}^*R\,R \equiv 0$ for this spacetime, the vanishing of the C Tensor leads to :

$$\partial_\phi \partial_t \theta = \partial_\vartheta \partial_t \theta = \partial_r \left(\frac{\partial_\phi \theta}{r}\right) = \partial_r \left(\frac{\partial_\vartheta \theta}{r}\right) = 0 \tag{18}$$

So, for a choice of $\theta = F(t, r) + rG(\phi, \vartheta)$, Eq. (17) will always be a solution.

Now we consider the following line element :

$$ds^2 = g_{\alpha\beta}\left(x^\gamma\right) dx^\alpha dx^\beta + \Phi\left(x^\gamma\right)^2 d\Omega^2 \qquad (19)$$

Where $d\Omega$ is the volume element of S^2 with coordinates x^i and $g_{\alpha\beta}$ is a 2D metric tensor with coordinates x^γ.
For this line element :

- $R_{\alpha\beta}$ and R_{ij} are non-vanishing

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The vanishing of the C-Tensor now leads to :

$$\theta = F\left(x^\gamma\right) + \Phi\left(x^\gamma\right) G\left(x^i\right) \tag{20}$$

And by virtue of Birkhoff's theorem : *The only spherically symmetric solution of the Modified Equations, in the non-dynamical framework, is the Schwarzschild solution*

Next we consider the Kerr metric :

$$
\begin{aligned}
ds^2 &= -\left(1-\frac{2Mr}{\Sigma}\right)dt^2 - \frac{4Mar\sin^2\vartheta}{\Sigma}dt d\phi + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\vartheta^2 + \\
&+ \sin^2\vartheta\left(r^2+a^2+\frac{2Mra^2\sin^2\vartheta}{\Sigma}\right)d\phi^2
\end{aligned}
\tag{21}
$$

where, $\Sigma = r^2 + a^2\cos^2\vartheta$ and $\Delta = r^2 - 2Mr + a^2$. As we already noticed, the Potryagin Contraint is not satisfied. Namely :

$$
{}^*R\,R = -\frac{96M^2 r\cos\vartheta}{\Sigma^6}\left(r^2-3a^2\cos^2\vartheta\right)\left(3r^2-a^2\cos^2\vartheta\right) \tag{22}
$$

Therefore the Kerr Metric is *not* a solution!

The γ-metric has the following line element, in prolate spheroidal coordinates :

$$
\begin{aligned}
ds^2 = & -\left(\frac{x-m}{x+m}\right)^{\gamma} dt^2 + \left(\frac{x^2-m^2}{x^2-m^2y^2}\right)^{\gamma^2} \left(\frac{x+m}{x-m}\right)^{\gamma} \times \\
\times & \left(x^2-m^2y^2\right)\left(\frac{dx^2}{x^2-m^2}+\frac{dy^2}{1-y^2}\right) + \\
+ & \left(\frac{x+m}{x-m}\right)^{\gamma} \left(x^2-m^2\right)\left(1-y^2\right) d\phi^2
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This metric :

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This metric :

- is static and axisymmetric
- Represents a Minkowski spacetime if $m = 0$ or $\gamma = 0$

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- is of Petrov type D - for $\gamma \neq 1$ and $\gamma \neq 0$

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- Represents a Minkowski spacetime if $m = 0$ or $\gamma = 0$
- Represents a Schwarzschild spacetime if $\gamma = 1$
- Represents a Curzon spacetime if $\gamma \to \infty$
- is of Petrov type D - for $\gamma \neq 1$ and $\gamma \neq 0$
- is the static limit of the Tomimatsu-Sato family of solutions

Next we will consider the following choice for the coupling field θ :

$$\theta = \theta(x) \tag{23}$$

The only non-vanishing component of the C-Tensor is :

$$C^{\phi t} = \left[\left(x^2 - m^2\right)\left(1-\gamma^2\right)\theta'' + (2x - m\gamma)\left(1-\gamma^2\right)\theta'\right]\frac{B(x,y)}{A(x,y)} \tag{24}$$

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this yields the following differential equation for θ :

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This equation has the following solution :

$$\theta = \left(\frac{x - m}{x + m}\right)^{\gamma/2} \tag{26}$$

To sum up :

- The only Spherically Symmetric Solution is the Schwarzschild Metric
- The Kerr Metric is *not* a solution of the Modified Field Equations
- The γ Metric is a solution of the Modified Field Equations for a specific choice of the coupling field θ

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- The γ Metric is a solution of the Modified Field Equations for a specific choice of the coupling field θ
- No other, axisymmetric (static or stationary), physically significant solutions have been found, yet.

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Conclusion : The overconstraining of the modified field equations, due to the *Pontryagin Contraint*, renders the existence of other solutions highly unlikely.

Vacuum Solutions in the Dynamical Framework

The Weyl line element in prolate spheroidal coordinates is :

$$
\begin{aligned}
ds^2 &= -f dt^2 + e^{2k} f^{-1} \left(x^2 - y^2\right) \left(\frac{dx^2}{x^2 - 1} + \frac{dy^2}{1 - y^2} \right) + \\
&+ \frac{\left(x^2 - 1\right)\left(1 - y^2\right)}{f} d\phi^2
\end{aligned}
\tag{27}
$$

Where f and k are functions of x and y only.

The Weyl line element in prolate spheroidal coordinates is :

$$ds^2 = -f\,dt^2 + e^{2k} f^{-1} \left(x^2 - y^2\right) \left(\frac{dx^2}{x^2-1} + \frac{dy^2}{1-y^2} \right) + \\ + \frac{\left(x^2-1\right)\left(1-y^2\right)}{f} d\phi^2 \qquad (27)$$

Where f and k are functions of x and y only.
Eq. (27) satisfies ${}^{*}R\,R \equiv 0$ so, after setting $V(\theta) \equiv 0$, Eq. (2) yields :

$$\Box\theta = 0 \qquad (28)$$

In addition to Eq. (28) the field equations must also satisfy :

$$C_{ab} - 8\pi T^{\theta}_{ab} = 0 \tag{27}$$

This is only true if :

$$\theta = \theta_1(x, y, t) + \theta_2(x, y, \phi) \tag{28}$$

so that the field equations will decouple.
In this case, Eq. (27) along with Eq. (28) lead to a very complex system of PDEs, that is impossible to solve!

We can describe such spacetimes with the Papapetrou line element (in cylindrical coordinates) :

$$ds^2 = -e^{2U}\left(dt - \omega d\phi^2\right)^2 + e^{-2U}\left[e^{2k}\left(d\rho^2 + dz^2\right) + \rho^2 d\phi^2\right] \tag{29}$$

where ω, U and k are all functions of ρ and z.
For such spacetimes we have, in general :

$${}^{*}R\,R \neq 0 \tag{30}$$

This leads to the following equation for θ :

$$\beta\Box\theta = -\frac{\alpha}{4}\,{}^{*}R\,R \tag{31}$$

This equation is a non-homogenous version of Eq. (28) and therefore constrains our system of PDEs even more since we also have to solve Eq. (27)

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We conclude that *no static or stationary axisymmetric solution of General Relativity can also be a solution of Chern-Simons Gravity.*