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A parallel code for multiprecision computations of the Lane–Emden differential equation

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Abstract: We compute multiprecision solutions of the Lane–Emden equation. This differential equation arises when introducing the well-known polytropic model into the equation of hydrostatic equilibrium for a nondistorted star. Since such computations are time-consuming, we apply parallel programming techniques; thus the execution time is drastically reduced.

1 Multiprecision and Parallel Programming Environments

We compute multiprecision solutions of the Lane–Emden equation of stellar hydrodynamics by a code implementing the Runge–Kutta–Fehlberg method of fourth and fifth order (see e.g. [1], Sec. 2.1), working in the multiprecision environment of the "Fortran–90 Multiprecision System" (MPFUN90) developed by D. H. Bailey ([2, 3], and references therein) — available in http://crd-legacy.lbl.gov/∼dhbailey/mpdist/ and licensed under the Berkeley Software Distribution License found in that site. Such highly accurate solutions can be used for checking other numerical codes and prescribing a measure of their accuracy. Since multiprecision computations are time-consuming, we apply parallel programming techniques appropriate for multicore machines. The Open Multi-Processing (OpenMP, http//openmp.org/) is an Application Program Interface (API) supporting shared-memory parallel programming in C/C++ and Fortran.

2 Polytropic Models in Astrophysics

Using the polytropic equation of state ([4], Chapter IV, Eq. (1)) in the equations of hydrostatic equilibrium and then introducing the normalization equations ([4], Chapter IV, Eqs. (8a), (10a)), we obtain the Lane–Emden equation ([4], Chapter IV, Eq. (11))

$$\theta'' + \frac{2}{\xi}\,\theta' = -\theta^n, \tag{1}$$

which, when integrated along a prescribed integration interval $\xi \in [\xi_{\rm start} = 0, \xi_{\rm end}] = \mathbb{I}_\xi \subset \mathbb{R}$ with initial conditions $\theta(\xi_{\rm start}) = 1$, $\theta'(\xi_{\rm start}) = 0$, gives as solution the Lane–Emden function $\theta[\mathbb{I}_\xi \subset \mathbb{R}]$. Our aim in this study is to compute multiprecision solutions $\theta[\mathbb{I}_\xi]$ of this "initial value problem" (IVP). There are however two problems regarding Eq. (1). First, to remove the indeterminate form θ'/ξ at the origin, appearing in the left-hand side, we modify this denominator by adding a tiny quantity, ξ_0, to it, i.e. $\theta'/(\xi + \xi_0)$. Since ξ_0 is small, the initial conditions are valid at the starting point $\xi_{\rm start} + \xi_0 = \xi_0$ as well. Accordingly, the integration interval becomes $\xi \in [\xi_0, \xi_{\rm end}] = \mathbb{I}_{\xi 0} \subset \mathbb{R}$. Second, for values ξ greater than the first root ξ_1 of $\theta(\xi)$, $\xi > \xi_1$, the Lane–Emden function changes sign, $\theta(\xi) < 0$, and thus the term θ^n in Eq. (1) becomes undefined (raising a negative real number to a real power, e.g. $-0.1^{1.5}$, is not defined in $\mathbb{R}$). This undefined issue is removed by taking instead the real power of the absolute value of θ, $|\theta|^n$. Note that this "numerical trick" is appropriate only when interested in finding the first root ξ_1 of the function θ — it becomes inaccurate when searching for higher roots. As it is usual in numerical analysis, we proceed by transforming Eq. (1) into a system of two first-order differential equations, with the IVP under consideration having then the form

$\theta' = \eta, \quad \eta' = -\frac{2}{(\xi+\xi_0)}\,\eta - |\theta|^n, \quad \xi \in \mathbb{I}_\xi, \quad \theta(0) = 1, \quad \eta(0) = 0$. By solving this IVP, we can compute several significant physical characteristics of a stellar model with finite radius, i.e. $n \in [0,\ 5)$.

3 Code, Parallel Programming, and Computations

We develop a code for solving the polytropic IVP. This code consists of two parts. The task of the first part is to provide all computer cores with the required variables and parameters. The second part performs parallel numerical computations for several values of the polytropic index. We use the work-sharing constructs of OpenMP in order to share the numerical work and to activate the available computer cores. A decisive step in parallel programming is the demarcation of the shared memory by using data-sharing attribute clauses like `SHARED`, `PRIVATE` and `FIRSTPRIVATE`. In particular, we specify the local variables as `PRIVATE`; so, each thread has its own copy of these variables. Variables having undefined values at the begging of the scope are declared as `FIRSTPRIVATE` within the shared region of the code. The computations are distributed over the computer cores by the `SCHEDULE(DYNAMIC)` clause. So, once a particular core finishes its allocated iteration, it returns to get another one from the iterations that are left. We use the worksharing construct `DO` in order to share the work of computing the first root for each polytropic index by a seperate computer core. The user has to provide an appropriate value to the integer variable `NMODEL` according to the number of computer cores available; thus the number of polytropic models which are being processed in parallel must be less or equal to `NMODEL`. We find the first root of the Lane–Emden function θ by combining `DDRKF54` (the modification of `DCRKF54` developed and used in [1] for solving complex IVPs) with a code that mimics the well-known bisection algorithm. The computations are performed in high-precision environment by MPFUN90. In this work, our computations are carried out with a presicion of 64 digits. Intergration takes place successively in two intervals. The first one is a very short interval $\mathbb{I}_1 = [10^{-26},\ 30 \times 10^{-26}]$ in order to accurately initiate the code `DDRKF54`. The second integration interval $\mathbb{I}_2$ extends from the end of $\mathbb{I}_1$ up to a value near $3\,\xi_1/2$. Some of the multiprecision results of this study are given in Table 1. A polytropic index appropriate for verifying the accuracy of our code is $n = 1$, since, as it is well-known, this case has an analytic solution and the first root of the Lane–Emden function is $\xi_1 = \pi$ ([4], Chapter IV, Eq. (45)).

Table 1: Constants of the Lane–Emden function.

n	ξ_1	$-\xi_1^2\theta'(\xi_1(n_\mathrm{s}))$
0.00	2.44948 97427 83178 09819 72840 74705 9	4.89897 94855 66356 19639 45681 49411 8
1.00	3.14159 26535 89793 23846 26433 83279 5	3.14159 26535 89793 23846 26433 83279 5
1.50	3.65375 37362 19122 42460 90942 80459 2	2.71405 51201 08645 71902 45332 77696 4
2.00	4.35287 45959 46124 67697 35700 61526 1	2.41104 60120 96893 78364 84427 44671 4
2.45	5.23614 14048 69233 45598 24188 11476 5	2.20681 79791 33278 31713 43386 65923 3
2.50	5.35527 54590 10779 45990 93600 02973 6	2.18719 95655 17078 95321 95209 89736 0
3.00	6.89684 86193 76960 37545 45281 87123 1	2.01823 59509 66228 40281 28131 70057 9
3.23	7.91690 48976 05477 15893 80785 74290 1	1.95492 90412 23479 62411 38693 00245 9
3.50	9.53580 53442 44850 44410 47426 25789 3	1.89055 70934 43116 39390 85853 05853 5

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