ΙΝΣΤΙΤΟΥΤΟ ΑΣΤΡΟΦΥΣΙΚΗΣ

Magnetised fluids in Astrophysics: Techniques and Algorithms Introduction to RAMSES

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What we will cover in this lecture

- The MHD equations
- Discretisation of the equations
- MHD waves
- The Godunov scheme
- Constrained Transport
- Introduction to the RAMSES code

The MHD equations

What is Magnetohydrodynamics (MHD)?

- Magnetohydrodynamics describes the behaviour of a plasma in large spatial and temporal scales. Specifically:
- It describes spatial scales larger than the Debye length:

$$\lambda_D = \left(\frac{k_B T}{4\pi n e^2}\right)^{1/2} = 6.904 \cdot T^{1/2} n^{-1/2} \ cm$$

- and temporal scales larger than the plasma and the cyclotron frequencies:

$$\omega_{pe} = \sqrt{\frac{4\pi n_e e^2}{m_e}}, \quad \omega_{pi} = \sqrt{\frac{4\pi n_i Z^2 e^2}{m_i}}, \quad \omega_{ce} = \frac{eB}{m_e}$$

When can we use MHD?

- When the plasma is **macroscopically neutral**
- When particle collisions are so frequent that the plasma is in a state of thermodynamical balance, so that $T_i = T_e$. The collision frequency between ions and electrons can be expressed as:

$$\nu_{ei} = n_0 \sigma_c v \simeq \frac{n_0 \pi e^4}{\sqrt{m_e (k_B T_e)^3}}$$

  Which means that the MHD approximation is valid when the electron/ion temperatures are small, and/or the mean density is large.

- Here we will talk about ideal, classical MHD, which means that we will be ignoring relativistic effects, friction and all the sources or losses of energy in the plasma.

Characteristics of some physical systems

The mean free path of a particle (the mean distance a particle can travel until it encounters another particle)

$$l_f = u_{th}/u_{ei}$$

where u_{th} the thermal velocity of the charges and u_{ei} the collision frequency is another way to examine the MHD approximation

Plasma	n_e (cm^{-3})	T (K)	λ_D (cm)	l_f
Earth's ionosphere	10^6	10^3	0.21	~ 1km
Earth's magnetosphere	1	10^7	$6.9\ 10^3$	(electron gyroradius) ~5km
Solar wind	1	10^5	$2.18\ 10^3$	~ 1AU
Ionised interstellar medium	10^{-1}	10^4	$2.34\ 10^3$	~ 0.01pc
Intergalactic medium	10^{-6}	10^6	$6.9 10^6$	~ 13 Mpc

Which of these systems can we describe with MHD?

Solar corona ($n \sim 10^{10}$ cm^{-3}, $T \sim 10^{6}$ K, $l_f \sim 8$ Rsun)

Image credit: P. Horálek/ESO

Cosmic rays ($n \sim 10^{-8}$ cm^{-3}, $l_f \sim$ 1AU-1kpc)

Image credit: Daniele Boffelli

The warm interstellar medium ($n \sim 10^{-2}$ cm^{-3}, $T \sim 8000$ K, $l_f = 20$km)

Image credit: NASA/JWST

Molecular clouds ($n \sim 10^{4}$ cm^{-3}, $T \sim 10$ K, $l_f = 20$km)

Image credit: NASA/JWST

GALAXIES ARE MAGNETISED

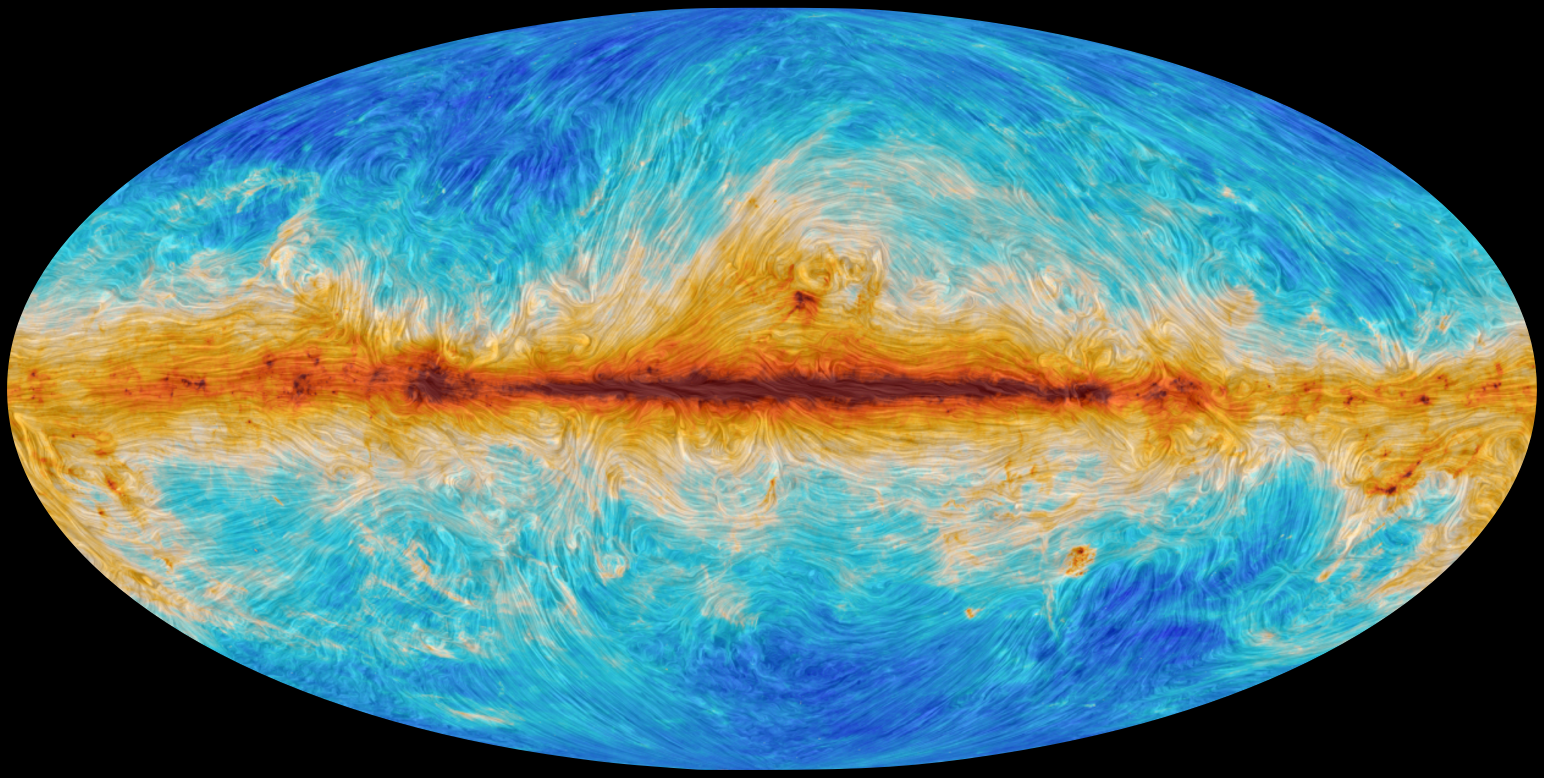
M51 from Serrano-Borlaff et al. 2021

Left: Magnetic field from FIR polarisation (this work)

Right Magnetic field from radio polarisation (Fletcher et al.2011)

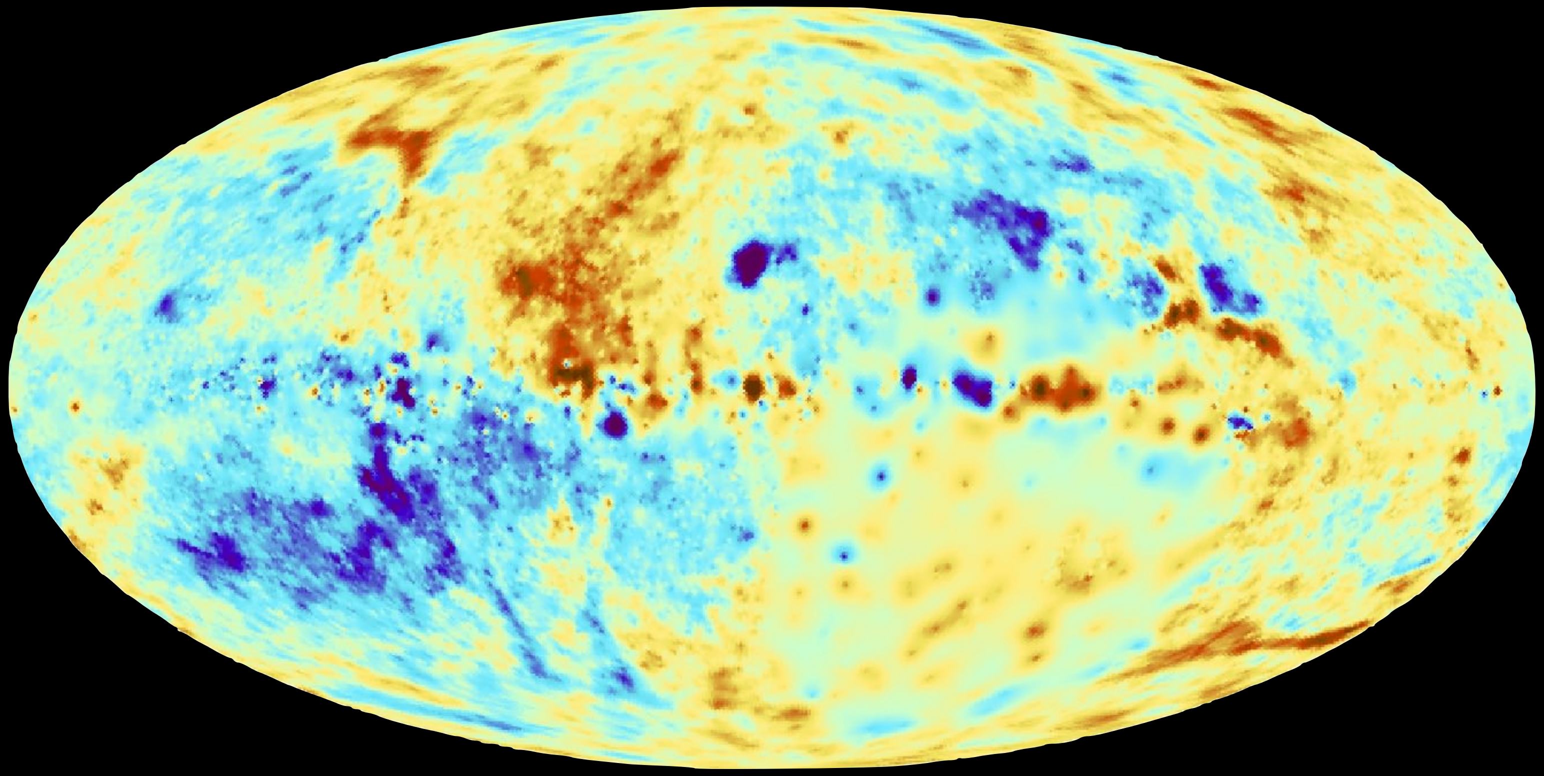
THE MAGNETIC FIELD OF THE MILKY WAY

Points to the field being dynamically important (e.g. Planck XXXII, 2014)



PLANCK DUST POLARIZATION MAP
FROM HTTPS://WWW.IAS.U-PSUD.FR/SOLER/PLANCKHIGHLIGHTS.HTML

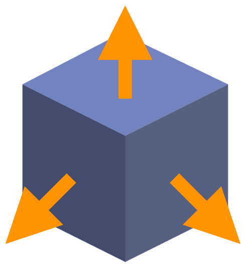
Shows a quadrupole in high latitudes and a coherent azimuthal field in the disk



FARADAY ROTATION SKY FROM OPPERMANN ET AL. 2012

The MHD equations in conservative form

1. Conservation of mass: $$\frac{\partial \rho}{\partial t} + \nabla \cdot \left[\rho \mathbf{u}\right] = 0$$



2. Momentum conservation: $$\frac{\partial\left(\rho \mathbf{u}\right)}{\partial t} + \nabla \cdot \left[\rho \mathbf{u}\mathbf{u} - \mathbf{B}\mathbf{B} + \mathbf{P}^*\right] = 0$$ $$P^* = P_{th} + B^2/2$$

3. Energy conservation: $$\frac{\partial E}{\partial t} + \nabla \cdot \left[(E + P^*)\mathbf{u} - \mathbf{B}(\mathbf{B}\cdot \mathbf{v})\right] = 0$$ $$E = \rho \mathbf{u}^2/2 + e + B^2/2$$

4. Flux conservation: $$\frac{\partial \mathbf{B}}{\partial t} = -c\nabla \times \mathbf{E}$$ $$\nabla \cdot \mathbf{B} = 0$$

Maxwell's equations

+ Equation of state

Ohm's law: $$\mathbf{J} = \sigma\left(\mathbf{E} + \frac{\mathbf{u}\times\mathbf{B}}{c}\right)$$

$$\frac{\partial \mathbf{B}}{\partial t} - \nabla \times (\mathbf{u}\times\mathbf{B}) = 0$$

For a fully conducting plasma: $\sigma \to \infty$ $\quad c\mathbf{E} = \mathbf{u}\times\mathbf{B}$

The MHD equations in non-conservative form

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0$$

$$\frac{\partial (\rho \mathbf{u})}{\partial t} + \nabla \cdot (\rho \mathbf{u}\mathbf{u}) = -\nabla P_{th} + \frac{1}{4\pi} \nabla \times (\nabla \times \mathbf{B})$$

$$\frac{\partial e}{\partial t} + \nabla \cdot (e \mathbf{u}) = -\frac{p}{\rho} \nabla \cdot \mathbf{u}$$

$$\frac{\partial \mathbf{B}}{\partial t} - \nabla \times (\mathbf{u} \times \mathbf{B}) = 0$$

+ Equation of state

Equation of state

To close the system we need **an equation of state**: $P=f(\rho)$

We usually adopt the ideal gas law:
$$P=nkT$$

In thermal equilibrium, each internal degree of freedom has energy $kT/2$.
Thus, the internal energy density for an ideal gas with m internal degrees of freedom is
$$e = nm(kT/2)$$

Combining the above:
$$P = (\gamma-1)e \text{ , where } \gamma = (m+2)/m$$

- For monoatomic gas (H), $\gamma=5/3$ ($m=3$)
- For diatomic gas (H_2), $\gamma=7/5$ ($m=5$)

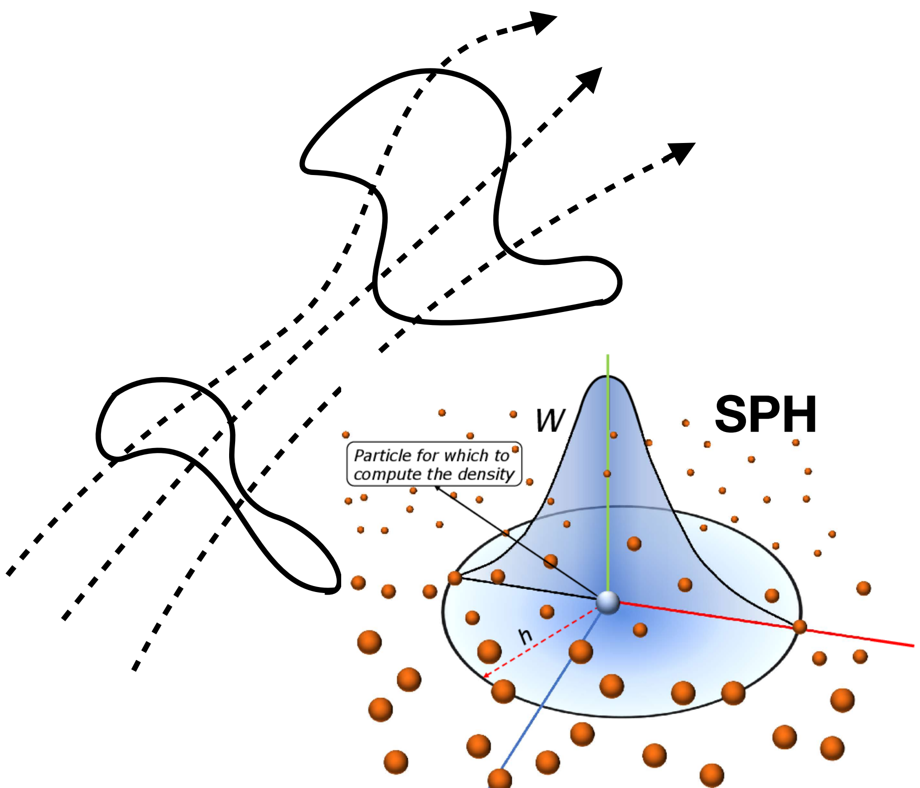
Also, when the radiative cooling time << the dynamical time
it is common to use the **isothermal EOS** $P = c_{sound}^2$ where c_{sound} is the isothermal sound speed.

In some circumstances, an ideal gas law is not appropriate, and must use more complex (or tabular) EOS (e.g. for degenerate matter)

Solving the MHD equations numerically

Approaches to solving the MHD equations

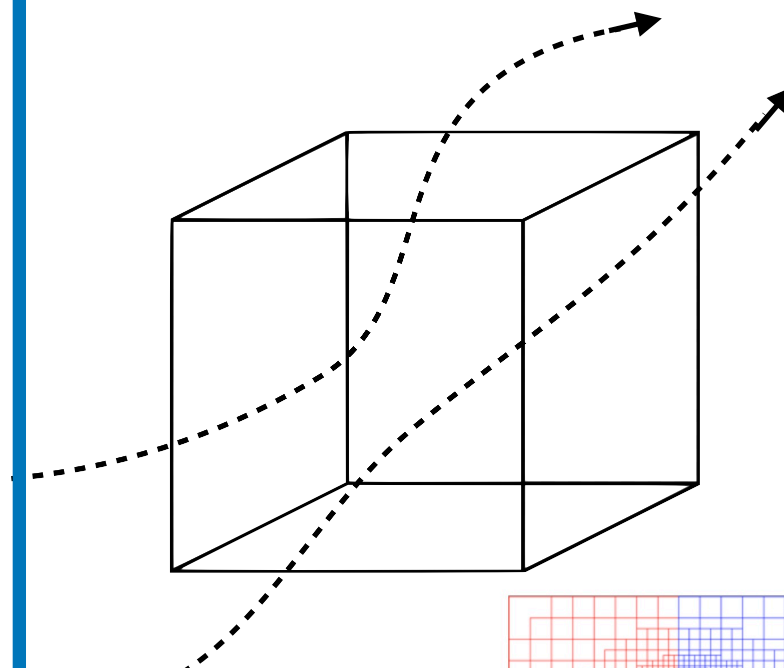
Lagrangian: Moving volume element

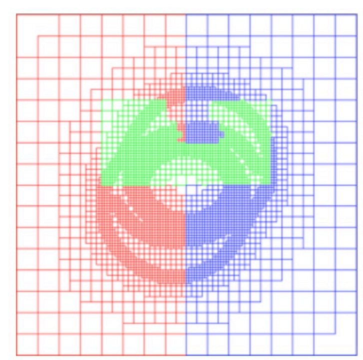


B'edorf & Zwart 2019

Gadget
SPHENIX
Bonsai-SPH
…

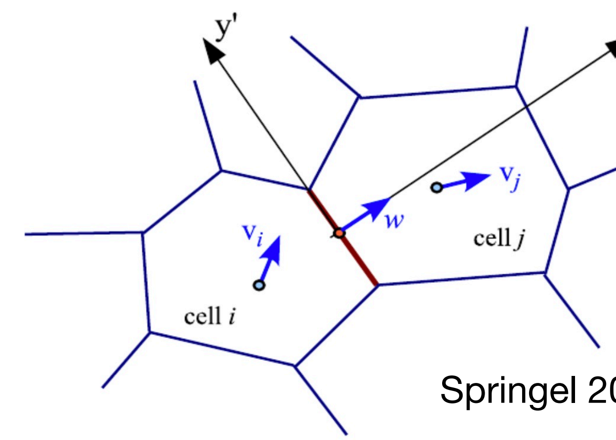
Eulerian: Static volume element



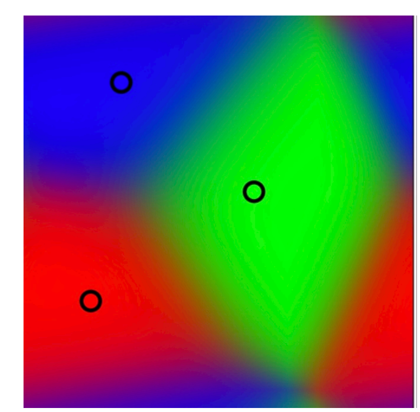


RAMSES
Athena++
FLASH
…

Hybrid (moving-mesh, Meshless Finite Volume)



Springel 2010



Hopkins 2015

AREPO
GIZMO

Lagrangian derivative → $$\frac{D}{Dt} = \frac{\partial}{\partial t} + \nabla \cdot \mathbf{u}$$ ← Eulerian derivative

Some basic concepts:

Discretisation: In Eulerean methods, we divide the computational space of length L in a number of cells: N_x, N_y, N_z per direction, so that we have a resolution of $\Delta x=L/N_x$, $\Delta y=L/N_y$, $\Delta z=L/N_z$

Round-off error

Not all real numbers can be represented by the machine.

If floating point operations result in a number that cannot be represented, some sort of rounding must be used.

It can be shown that the relative error can be bounded by a very small number ε, called "machine precision":

$$\frac{|x - x'|}{|x|} < \epsilon$$

Truncation error

Numerical algorithms approximate the true (analytic) solution.

The difference between the true and the approximate solution is called truncation error.

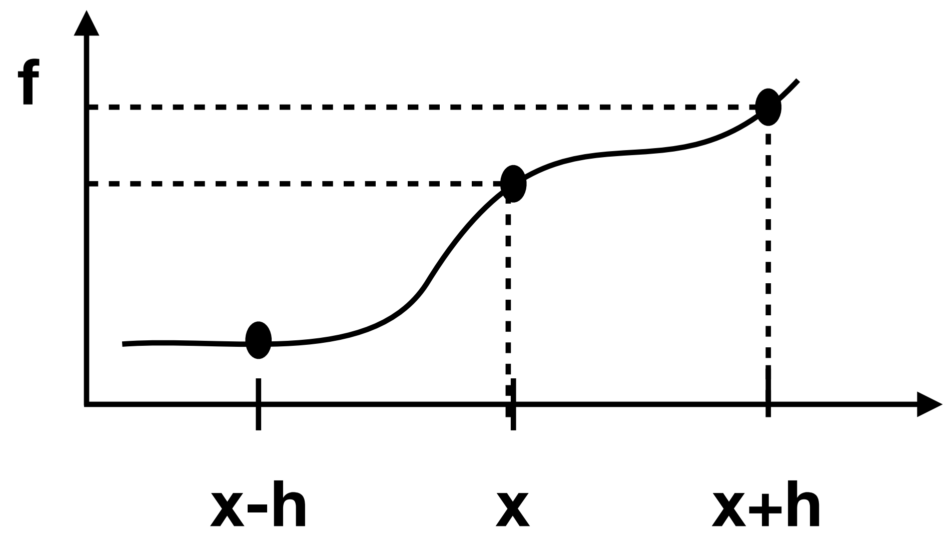
The TE is not related to the finite precision of numbers on a computer (round-off error).

Consistency: truncation error must decrease as resolution increased.

Convergence: numerical solution should approach analytic solution as grid spacing Δx decreases (numerical resolution increases).

Stability: round-off error must remain small and bounded.

Simple finite differences



The obvious approximation to a derivative is taking the difference:

$$f'(x) \simeq \frac{f(x+h) - f(x)}{h}$$

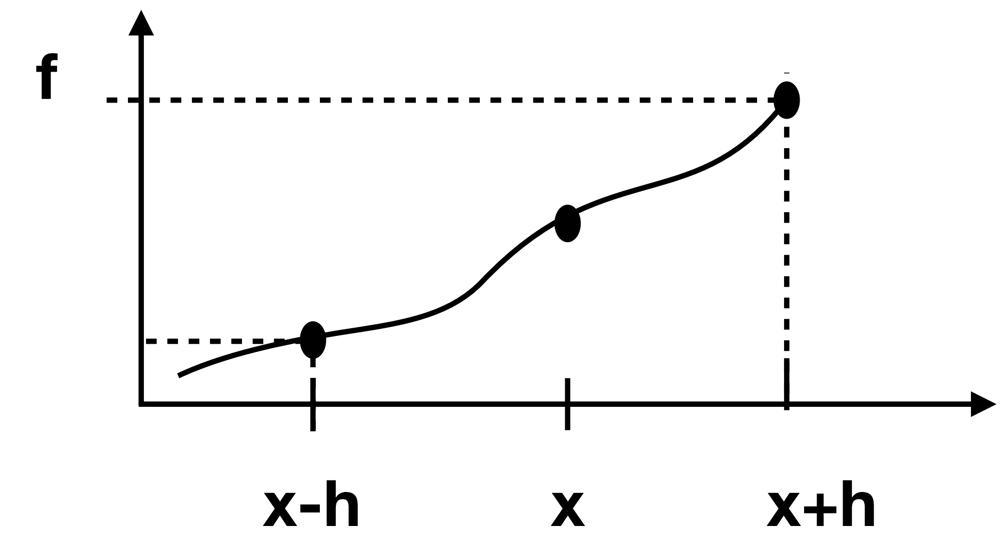
Here, the leading term in the truncation error is O(h), so the method is "first-order accurate" (think of Taylor series expansion).

In addition to the truncation error, there will be round-off error in evaluating the derivative:

- when x >> h, x+h inaccurate
- when evaluating f(x), the error is magnified.

Round-off error of this simple form for the derivative is **at best** sqrt(ε)

Centered differences



A significantly more accurate approach is taking a centred difference:

$$f'(x) \simeq \frac{f(x+h) - f(x-h)}{2h}$$

In this case, the leading term dropped in the Taylor expansion is $O(h^2)$, which can lead to order of magnitude decrease in truncation error.
We can get a second-order scheme almost for free!

Using Taylor series, it is also easy to build successively higher-order approximations for the derivative (careful, **only for smooth functions**!) e.g. PENCIL code (Pencil Code Collaboration 2021).

However, the solution can still be unstable when there are shocks!
Oscillations near discontinuities often appear, and methods can easily go catastrophically unstable:
We need to account for the physics of the discontinuity

and we know there are some shocks in the ISM!

Hyperbolic conservation laws

The MHD equations:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot [\rho \mathbf{u}] = 0$$

$$\frac{\partial (\rho \mathbf{u})}{\partial t} + \nabla \cdot [\rho \mathbf{u}\mathbf{u} - \mathbf{B}\mathbf{B} + \mathbf{P}^*] = 0$$

$$\frac{\partial E}{\partial t} + \nabla \cdot [(E + P^*)\mathbf{u} - \mathbf{B}(\mathbf{B} \cdot \mathbf{v})] = 0$$

$$\frac{\partial \mathbf{B}}{\partial t} - \nabla \times (\mathbf{u} \times \mathbf{B}) = 0$$

can be written in the form $\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}}{\partial x} = 0$

For a 1D problem:
$\nabla \cdot \mathbf{B} = 0 \Rightarrow B_x = 0$

Where $\mathbf{U} = \begin{pmatrix} \rho \\ \rho u_x \\ \rho u_y \\ \rho u_z \\ B_y \\ B_z \\ E \end{pmatrix}$ and $\mathbf{F} = \begin{pmatrix} \rho u_x \\ \rho u_x^2 + P^* - B^2 \\ \rho u_x u_y - B_x B_y \\ \rho u_x u_z - B_x B_z \\ B_y u_x - B_x u_y \\ B_z u_x - B_x u_z \\ (E + P^*) u_x - B_x (B_x u_x + B_y u_y + B_z u_z) \end{pmatrix}$

We can rewrite as

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}}{\partial \mathbf{U}} \frac{\partial \mathbf{U}}{\partial x} = 0$$

which is a hyperbolic PDE

If $\frac{\partial \mathbf{F}}{\partial \mathbf{U}}$ is constant

we have a linear hyperbolic PDE

Hyperbolic PDE:

A hyperbolic Partial Differential Equation (PDE) of order n is a well-posed initial value problem for its first n-1 derivatives. The best example of a hyperbolic PDE is the wave equation:

$$\frac{\partial^2 u}{\partial t^2} = c_w^2 \frac{\partial^2 u}{\partial x^2}$$

Advection (entropy wave)

If P and u_x are constant, it is easy to find time-dependent solutions to the **hydro** equations representing advection (entropy wave).

$$\mathbf{U} = \begin{pmatrix} \rho \\ \rho u_x \\ \rho u_y \\ \rho u_z \\ E \end{pmatrix} \qquad \mathbf{F} = \begin{pmatrix} \rho u_x \\ \rho u_x^2 + P^* \\ \rho u_x u_y \\ \rho u_x u_z \\ (E + P^*) u_x \end{pmatrix} \qquad \frac{\partial \mathbf{F}}{\partial \mathbf{U}} = u_x$$

Fluid equations become

$$\frac{\partial \mathbf{U}}{\partial t} + u_x \frac{\partial \mathbf{U}}{\partial x} = 0$$

Which has solution: $f(x, t) = f(x - u_x t, 0)$

At any later time, the solution is just the initial condition displaced by $u_x t$.
In particular, the density field moves with flow without changing shape (advection).
Even discontinuous solutions are allowed for the density, which just move with flow (contact discontinuties).

Sound waves

Hyperbolic PDEs admit solutions of the form:

$$\alpha = \alpha_0 + a_1 \exp(i\mathbf{k}x - i\omega t)$$

WAVES

- When $\alpha_1/\alpha_0 << 1$; waves are small amplitude and remain linear
- When $\alpha_1/\alpha_0 > 1$, waves are large amplitude and will become nonlinear

We substitute the solution for plane waves into the hydrodynamic equations, assuming a uniform homogeneous background medium, so that α_0=constant and v0=0. We keep only linear terms, so the fluid equations become:

$$-i\omega\rho_1 = -i\rho_0\mathbf{k}\cdot\mathbf{u_1}$$

$$-i\omega\mathbf{v_1} = -i\frac{1}{\rho_0}\mathbf{k}P_1$$

$$-i\omega P_1 = -i\gamma P_0\mathbf{k}\cdot\mathbf{u_1}$$

Linear system with constant coefficients.
Requiring det(A)=0 yields:

$$\omega^3(\omega^2 - c_{sound}^2 k^2) = 0$$

$$c_{sound}^2 = \frac{\gamma P}{\rho}$$

5 modes: 3 advection modes and 2 sound waves with $\omega/k = \pm c_{sound}$

MHD waves

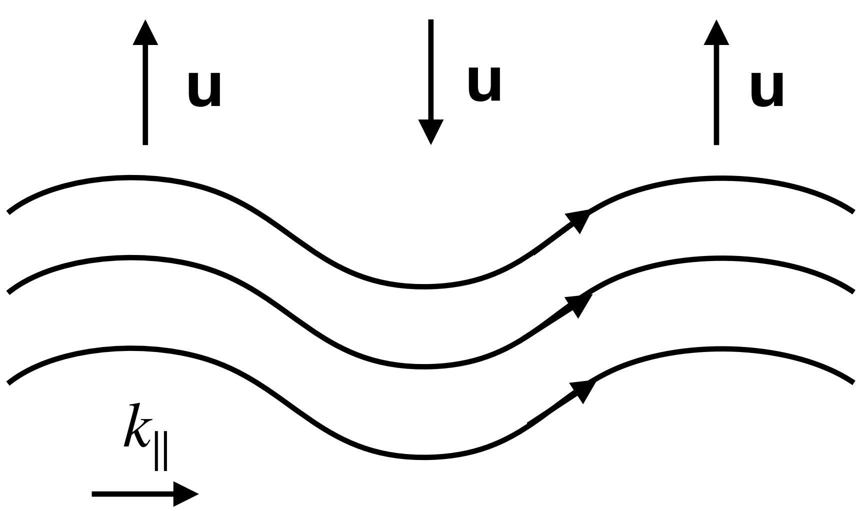
If we repeat the exercise of inserting the wave solution into the MHD equations, assuming a uniform homogeneous background medium, and keeping only linear terms, we get a much more complicated dispersion relation:

$$\left[\omega^2 - (\mathbf{k}\cdot\mathbf{c_a})^2\right]\left[\omega^4 - \omega^2 k^2 (c_a^2 + c_{sound}^2) + k^2 c_{sound}^2 \big(\mathbf{k}\cdot\mathbf{u_a})^2\right] = 0$$

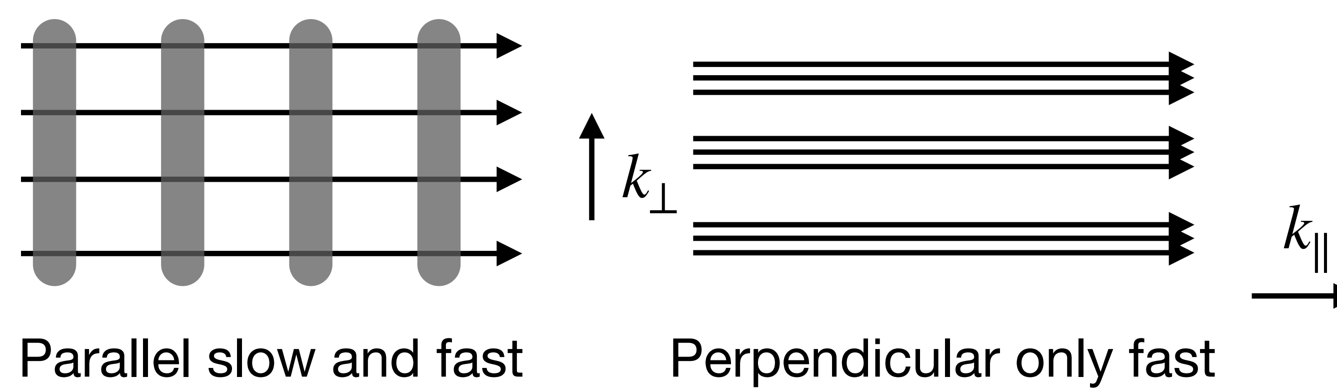
Now we have three modes instead of one:

- **Alfven wave** propagates at c_a
- **Slow and fast magnetosonic waves** propagating at c_{sound} and c_f

The entropy mode is also present in both cases!



Alfvén waves
Represent propagating transverse perturbations of the B-field.



Fast and slow magnetosonic
Compressible perturbations of both field and gas.
Fast mode has field and gas compression in phase
Slow mode has field and gas compression out of phase.

Summary of MHD waves

By calculating the 7 eigenvalues of the Jacobian matrix: $\mathbf{J} = \dfrac{\partial \mathbf{F}}{\partial \mathbf{U}}$

→ Propagation velocities of the various MHD waves

$$\lambda_a = u \pm c_a \qquad \lambda_e = u \qquad \lambda_s = u \pm c_s \qquad \lambda_f = u \pm c_f$$

$$c_a^2 = \frac{B^2}{4\pi\rho} \qquad\qquad c_{sound}^2 = \frac{\gamma P}{\rho}$$

Alfvén waves:
transverse waves with no variation in pressure and density

Entropy waves:
contact discontinuities with no variation in pressure and velocity

$$c_f^2 = \frac{1}{2}\left(c_{sound}^2 + c_a^2\right) + \sqrt{\left(c_{sound}^2 + c_a^2\right)^2 - 4c_{sound}^2 c_{a,x}^2}$$

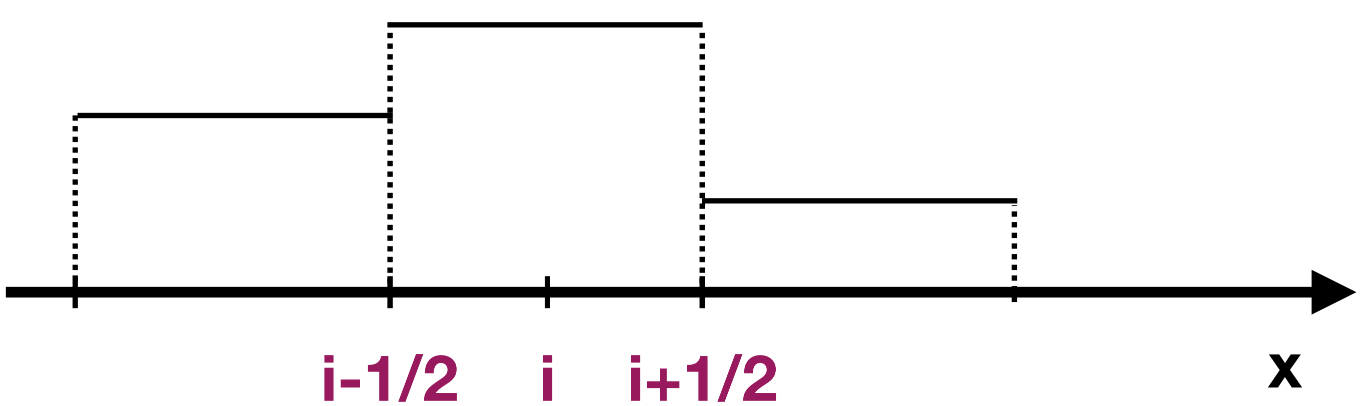
$$c_s^2 = \frac{1}{2}\left(c_{sound}^2 + c_a^2\right) - \sqrt{\left(c_{sound}^2 + c_a^2\right)^2 - 4c_{sound}^2 c_{a,x}^2}$$

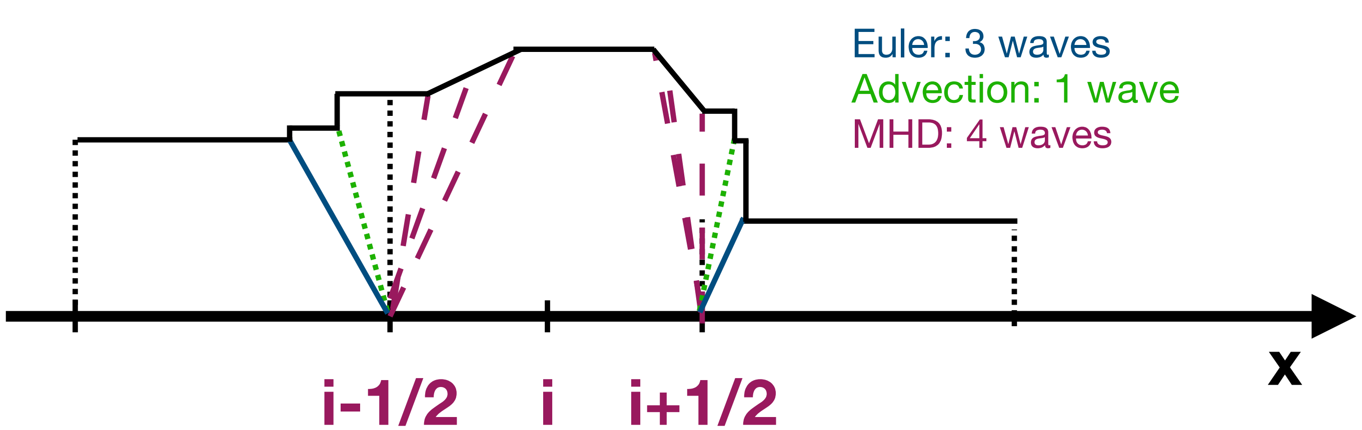
Fast magnetosonic waves:
longitudinal waves with variations in pressure and density
(correlated with magnetic field)

Slow magnetosonic waves:
longitudinal waves with variations in pressure and density
(anti-correlated with magnetic field)

Godunov scheme (first-order)

- The differences between cell-averaged values at each grid interface define a set of Riemann problems (evolution of initially discontinuous states).
- The solutions of these Riemann problems averaged over each cell give the time evolution of the cell-averaged values, until the MHD waves from one interface cross the cell and interact with the next one.
- This is called the "Courant-Friedrichs-Lewy" condition:
  Δt must be less than $\Delta x/(u+c)$.
- We don't actually need to solve the Riemann problem exactly. We just need to compute the state at the location of the interface in order to compute fluxes.





Godunov (1959)

The system of conservation laws is discretised as:

$$\frac{\mathbf{U}_j^{n+1} - \mathbf{U}_j^n}{\Delta t} = -\frac{\mathbf{F}_{i+1/2}^{n+1/2} - \mathbf{F}_{i-1/2}^{n+1/2}}{\Delta x}$$

Then the time-averaged flux function is computed using the self-similar solution of the inter-cell Riemann problem:

$$\mathbf{U}^*_{\mathbf{i+1/2}}(\mathbf{x/t = 0}) = \mathscr{R}\mathscr{P}(\mathbf{U^n_i}, \mathbf{U^n_{i+1}})$$

$$\mathbf{F^{n+1/2}_{i+1/2}} = \mathbf{F}(\mathbf{U}^*_{\mathbf{i+1/2}}(\mathbf{0}))$$

This defines the Godunov flux:

$$\mathbf{F^{n+1/2}_{i+1/2}} = \mathbf{F}^*(\mathbf{U^n_i}, \mathbf{U^n_{i+1}})$$

Riemann solvers

- For pure hydrodynamics of ideal gases, exact/efficient nonlinear Riemann solvers are possible.
- In MHD, nonlinear Riemann solvers are complex because:
  1. There are 3 wave families in MHD, so seven characteristics
  2. In some circumstances, 2 of the 3 waves can be degenerate (e.g. $c_a = c_{sound}$)
  3. The MHD equations are not strictly hyperbolic (Brio & Wu 1988, Zachary & Colella 1992)
- In practice, MHD Godunov schemes use approximate and/or linearised Riemann solvers.
- Since the solution of the Riemann problem only impacts the solution through the states of two neighbouring cells, we consider approximations where only one wave propagates in each direction (two-wave solvers).
- The solution will have an intermediate state $\mathbf{Q}_m = \dfrac{\mathbf{F}(\mathbf{Q_R}) - \mathbf{F}(\mathbf{Q_L}) - \mathbf{c_2}\mathbf{Q_R} + \mathbf{c_1}\mathbf{Q_L}}{c_1 - c_2}$

It remains only to specify the wave speeds, and it is in this specification that the various two-wave Riemann solvers differ.

The LLF (Local Lax-Friedrichs) Riemann solver

The Lax-Friedrichs method assumes that both waves have the same speed, c , in opposite directions. Then:

$$\mathbf{Q}_m = \frac{\mathbf{F}(\mathbf{Q_r}) - \mathbf{F}(\mathbf{Q_l})}{2c} - \frac{\mathbf{Q_r} + \mathbf{Q_l}}{2}$$

The local Lax-Friedrichs admits different values of c at different interfaces

For the method to be stable, one should choose the highest velocity of the Riemann solution across the boundary.

However, this causes significant diffusion, because it damps the slower wave.

The HLL-type Riemann solvers

HLL-type Riemann solvers rely only on computing the fastest wave speed.

Define c_f as the fast magnetosonic speed and the left and right going waves as:

$$u_L = min(u_L, u_R) - max(c_{f,L}, c_{f,R}) \qquad u_R = max(u_L, u_R) - max(c_{f,L}, c_{f,R})$$

Use generic Rankine-Hugoniot relations with one single intermediate state U*
and corresponding flux F*, we get the HLL flux:

$$u_L > 0 : \mathbf{F}^*(\mathbf{U_L}, \mathbf{U_R}) = \mathbf{F_L}$$

$$u_R < 0 : \mathbf{F}^*(\mathbf{U_L}, \mathbf{U_R}) = \mathbf{F_R}$$

$$u_L < 0 \; and \; u_R > 0 : \mathbf{F}^* = \frac{u_R F_L - u_l F_R + u_L u_R (u_R - u_L)}{u_R - u_L}$$

The Lax-Friedrich flux is obtained as a particular case with $u^* = u_R = -u_L$

The problem with the induction equation in more dimensions

The conservative form of the induction equation uses
the divergence of the Maxwell stress tensor:

$$\nabla\left(\frac{B^2}{2}\delta_{ij} - B_iB_j\right) + (\nabla\cdot\mathbf{B})\,\mathbf{B} = -\,\mathbf{J}\times\mathbf{B}$$

If **B** has no divergence, then the divergence of the Maxwell stress tensor equals the Lorenz force and we are fine.
BUT if magnetic monopoles are forming due to numerical truncation errors, the induction equation doesn't remove them. Then when we derive the induction equation, we have a spurious force:

$$\partial_t(\rho\mathbf{u}) + \nabla(\rho\mathbf{u}\mathbf{u} - \frac{\mathbf{1}}{\mathbf{4}\pi}\mathbf{B}\mathbf{B}) + \nabla(\mathbf{P}^*) = -\,(\nabla\cdot\mathbf{B})\mathbf{B}$$

Non-zero divergence accumulates, giving rise to a spurious force parallel to the field lines. In some cases, div B will grow without bounds (numerical instability)!

The challenge of computational MHD is to design divB preserving schemes.

The problem with the induction equation in more dimensions

Many schemes have been proposed to get rid of the divergence issue in cell-centered approaches:

- Powell's 8-wave scheme (Powell 1999) explicitly introduces magnetic monopole and magnetic current by adding source terms to the momentum equation and to the induction equation. So we have 8 characteristics, with a "divB" wave.

  The monopoles are advected away, but the jump conditions at the cell interfaces are incorrect.

- The projection scheme (Brackbil & Barnes 1980) computes the monopole (magnetic charge, m) for each cell and solves a Poisson equation $\Delta\Phi=m$, and use the gradient of Φ to correct the field.

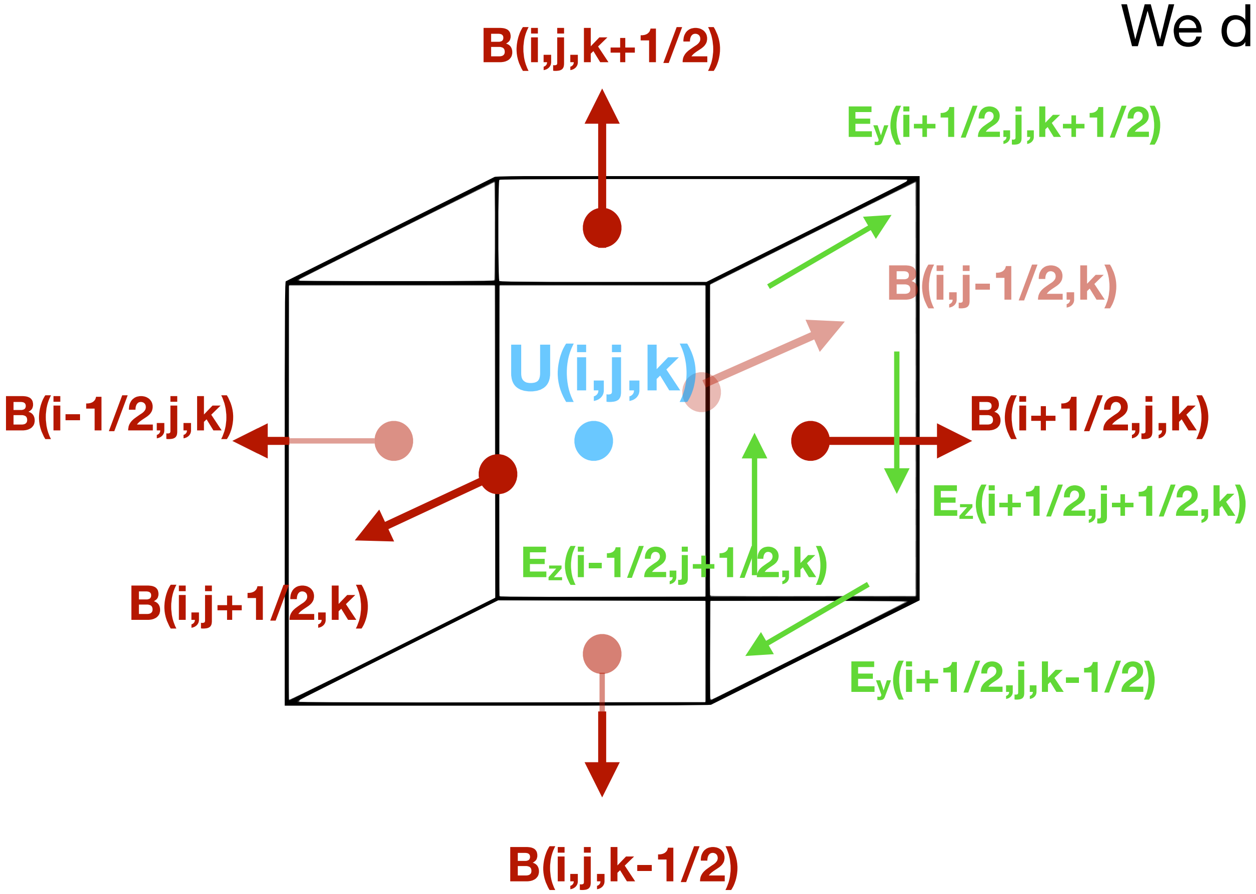
  The magnetic field is close to the true, but Poisson equation solving is time-consuming. Also, the correction causes perturbations in the pressure.

- Dedner's diffuson scheme (Dedner et al. 2002) is a variant of the projection scheme, with an hyperbolic div B cleaning step.
- Combination of the above (Crockett et al. 2005)

Constrained Transport

Yee (1966)
Evans & Hawley (1988)

$$\nabla \cdot \mathbf{B} = \lim_{\Delta V \to 0} \frac{1}{\Delta V} \oint_{S=\partial \Delta V} \mathbf{B} \cdot \mathbf{n} \, dS$$



We define the magnetic flux on cell faces, for example:

$$\Phi(i+1/2, j, k) = \frac{1}{S} \int B_x(y,z) dy dz$$

$$S = [y_{i-1/2}, y_{i+1/2}] \times [z_{i-1/2}, z_{i+1/2}]$$

And take a line integral of the electric field along cell edges, for example:

$$\mathscr{E}_{i+1/2,j+1/2,k} = -\int_{z(k-1/2)}^{z(k+1/2)} E_z(x(i+1/2), y(j+1/2)) dz$$

The magnetic flux at each face is updated from the circulation of the electric field:

$$\frac{d\Phi_{i+1/2,j,k}}{dt} = \mathscr{E}_{i+1/2,j+1/2,k} - \mathscr{E}_{i+1/2,j-1/2,k} - \mathscr{E}_{i+1/2,j,k+1/2} + \mathscr{E}_{i+1/2,j,k-1/2}$$

Cell-centered mass, momentum, energy
but: face-centered magnetic field

Each of the line integrals of the electric field is shared by two faces, but appears with opposite sign in the time update formula
So the total flux (div B) across each cell bounding surface vanishes exactly!

Advantages/disadvantages of each method

Lagrangian: Moving volume element	Eulerian: Static volume element	Hybrid (moving-mesh)
Smears out shocks and discontinuities	Riemann solvers are great for capturing shocks!	Riemann solvers are great for capturing shocks!
Hard to implement $\nabla \cdot \mathbf{B} = 0$	Easy to implement $\nabla \cdot \mathbf{B} = 0$	Hard to implement $\nabla \cdot \mathbf{B} = 0$
Naturally Galilean-invariant	Truncation errors depend on velocities	Naturally Galilean-invariant

Choose a code according to the needs of your problem!

The RAMSES code

(Teyssier 2002, Fromang et al. 2006)

Features of the code:

- High-order Godunov scheme

The original, first-order Godunov scheme is very diffusive
van Leer (1979) developed a second-order Godunov method
where the flux function is approximated using a predictor-corrector scheme

- Various Riemann solvers for MHD, including LLF and HLLD

- Staggered mesh for the magnetic field with Constrained Transport

Features of the code we didn't have time for today:

- Adaptive Mesh Refinement
- MPI parallelisation scheme
- Gravity (Poisson) solvers
- Particle Mesh for collisionless particles
- Sources/losses of energy
- Non-ideal MHD
- Radiation
- Sub-grid models

Ready to run some MHD simulations?

Credit: A. Konstantinou

Credit: P. Hennebelle