The 5th Summer School of Hel.A.S.
“Magnetohydrodynamics in Astrophysics”
Problem solving in MHD

Problem 1

Assume a thin vertical sheet of magnetized plasma in which the pressure $P(x)$, density $\rho(x)$, and vertical magnetic field $B_z(x)$ depend only on the horizontal coordinate x perpendicular to the sheet. The temperature, T, and horizontal magnetic field components, B_x and B_y, are constant. Assume also steady state (i.e. $\partial/\partial t = 0$) and that the influence of gravity to the MHD momentum equation is non-negligible.

(a) Prove that the x and z components of the MHD momentum equation reduce to

$$-\frac{\partial}{\partial x}\left(P + \frac{B^2}{8\pi}\right) = 0 \tag{1}$$

and

$$-\rho g + \frac{B_x}{4\pi}\frac{\partial B_z}{\partial x} = 0 \tag{2}$$

respectively, where g is the acceleration of gravity.

(b) By defining the boundary conditions far away from the sheet, $P(x \to \pm\infty) = 0$ and $B_z(x \to \pm\infty) = \pm B_{z\infty}$, show that

$$P(x) = \frac{B_{z\infty}^2 - B_z^2(x)}{8\pi} \tag{3}$$

(c) By using the equation of state for ideal gases, the mass density, and the definition of the pressure scale height, λ_p, show that from the above equations we obtain

$$-\frac{B_{z\infty}^2 - B_z^2(x)}{2\lambda_p} + B_x\frac{\partial B_z(x)}{\partial x} = 0 \tag{4}$$

(d) To save some time, do NOT attempt to solve equation (4). Use that its solution is

$$B_z(x) = B_{z\infty}\tanh\left(\frac{B_{z\infty}}{B_x}\frac{x}{2\lambda_p}\right) \tag{5}$$

and find that for the pressure $P(x)$ we obtain

$$P(x) = \frac{B_{z\infty}^2}{8\pi}\left[\operatorname{sech}\left(\frac{B_{z\infty}}{B_x}\frac{x}{2\lambda_p}\right)\right]^2 = \frac{B_{z\infty}^2}{8\pi}\left[\cosh\left(\frac{B_{z\infty}}{B_x}\frac{x}{2\lambda_p}\right)\right]^{-2} \tag{6}$$

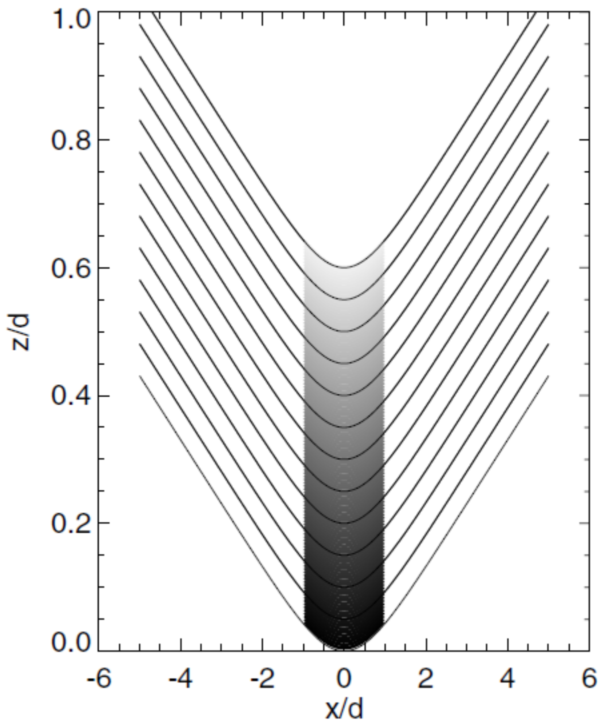

Figure 1: Computed magnetic field lines of the model described in Problem 1. The grey area marks the vertical plasma material.

(e) The magnetic field lines of the above model can be computed using $dx/B_x = dz/B_z$. The result is presented in Figure 1. By considering the various forces exerted on the vertical plasma sheet, discuss its stability.

FORMULARY

Equations of ideal MHD: $\dfrac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{v}) = 0$, $\rho \dfrac{\partial v}{\partial t} = -\boldsymbol{\nabla} P + \dfrac{\boldsymbol{J} \times \boldsymbol{B}}{c}$, $\boldsymbol{E} + \dfrac{\boldsymbol{v} \times \boldsymbol{B}}{c} = \eta \boldsymbol{J}$, $\boldsymbol{\nabla} \times \boldsymbol{B} = \dfrac{4\pi}{c}\boldsymbol{J}$, $\boldsymbol{\nabla} \times \boldsymbol{E} = -\dfrac{1}{c}\dfrac{\partial \boldsymbol{B}}{\partial t}$.

Equation of state for ideal gases: $P = 2n_e kT$

Mass density: $\rho = mn = \mu m_H n_e$

Pressure scale height: $\lambda_p = \dfrac{2kT}{\mu m_H g}$

Vector identity: $\boldsymbol{\nabla}(\boldsymbol{a} \cdot \boldsymbol{b}) = (\boldsymbol{a} \cdot \boldsymbol{\nabla})\boldsymbol{b} + (\boldsymbol{b} \cdot \boldsymbol{\nabla})\boldsymbol{a} + \boldsymbol{a} \times (\boldsymbol{\nabla} \times \boldsymbol{b}) + \boldsymbol{b} \times (\boldsymbol{\nabla} \times \boldsymbol{a})$.