MHD Instabilities

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Outline

- linear (normal mode) stability analysis:
  methodology to find growth rates and eigenfunctions based on a single “principal” differential equation

- notable examples of classical instabilities (Rayleigh-Taylor, Kelvin-Helmholtz, and the influence of magnetic field on them, current-driven)

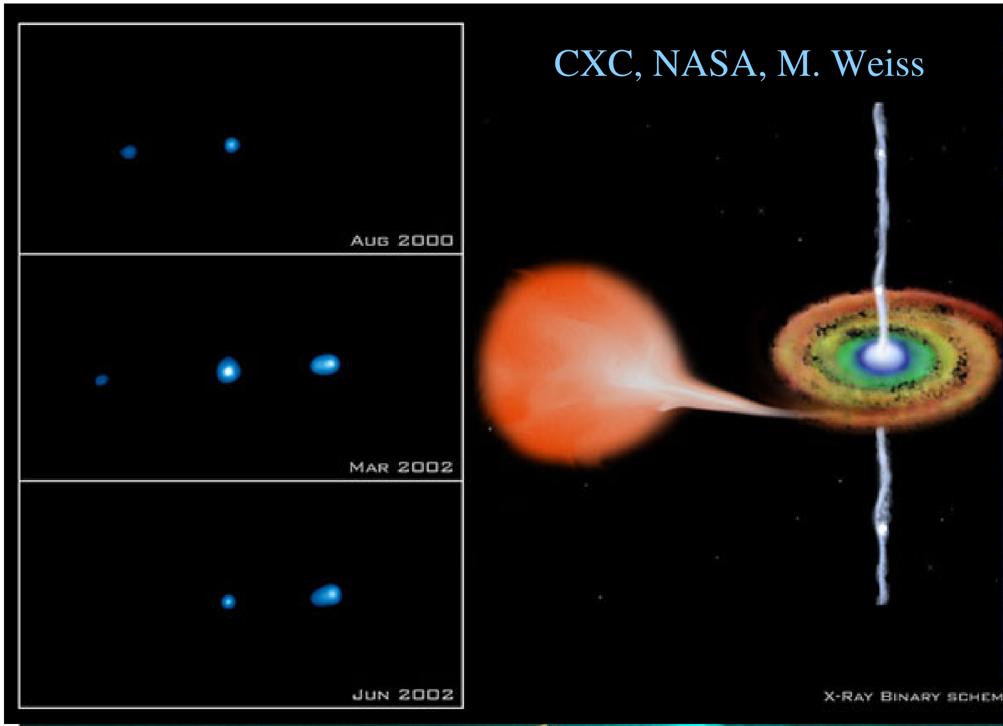
Why study MHD instabilities

- to understand the evolution of perturbations is essential in all kinds of equilibria in Physics (to know if a given equilibrium is stable and the period of small oscillations – to know the growth time if it is unstable)

- instabilities are important in many Astrophysical processes (cloud collapse, disk accretion, generation of turbulence / magnetic reconnection / energy dissipation / particle acceleration)

- magnetic fields could have important (or even the dominant) role in these instabilities

- may help to solve the thermonuclear fusion problem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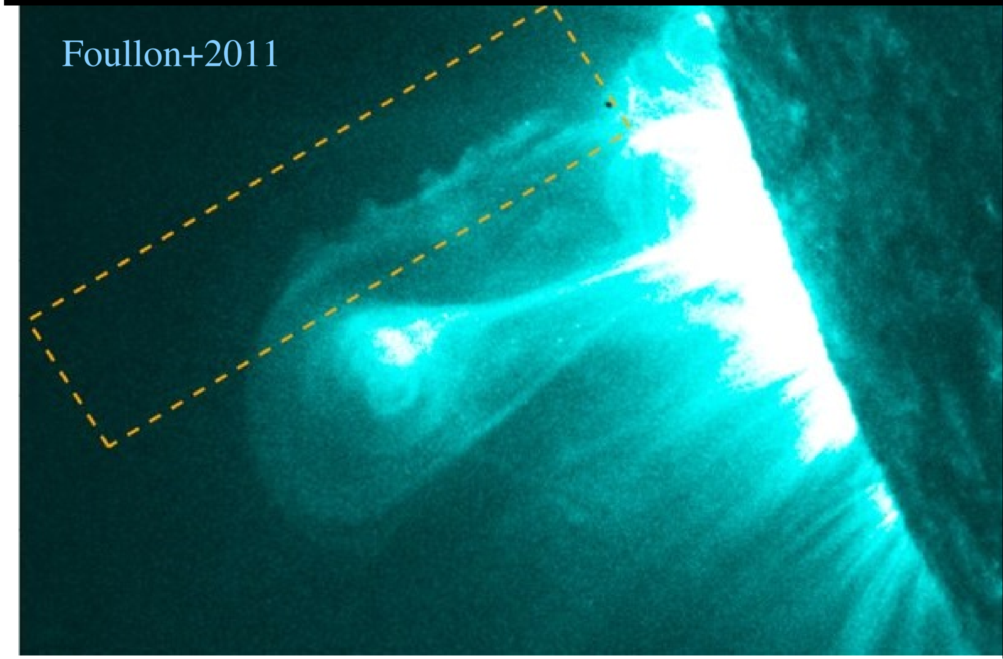
X-RAY BINARY SCHEM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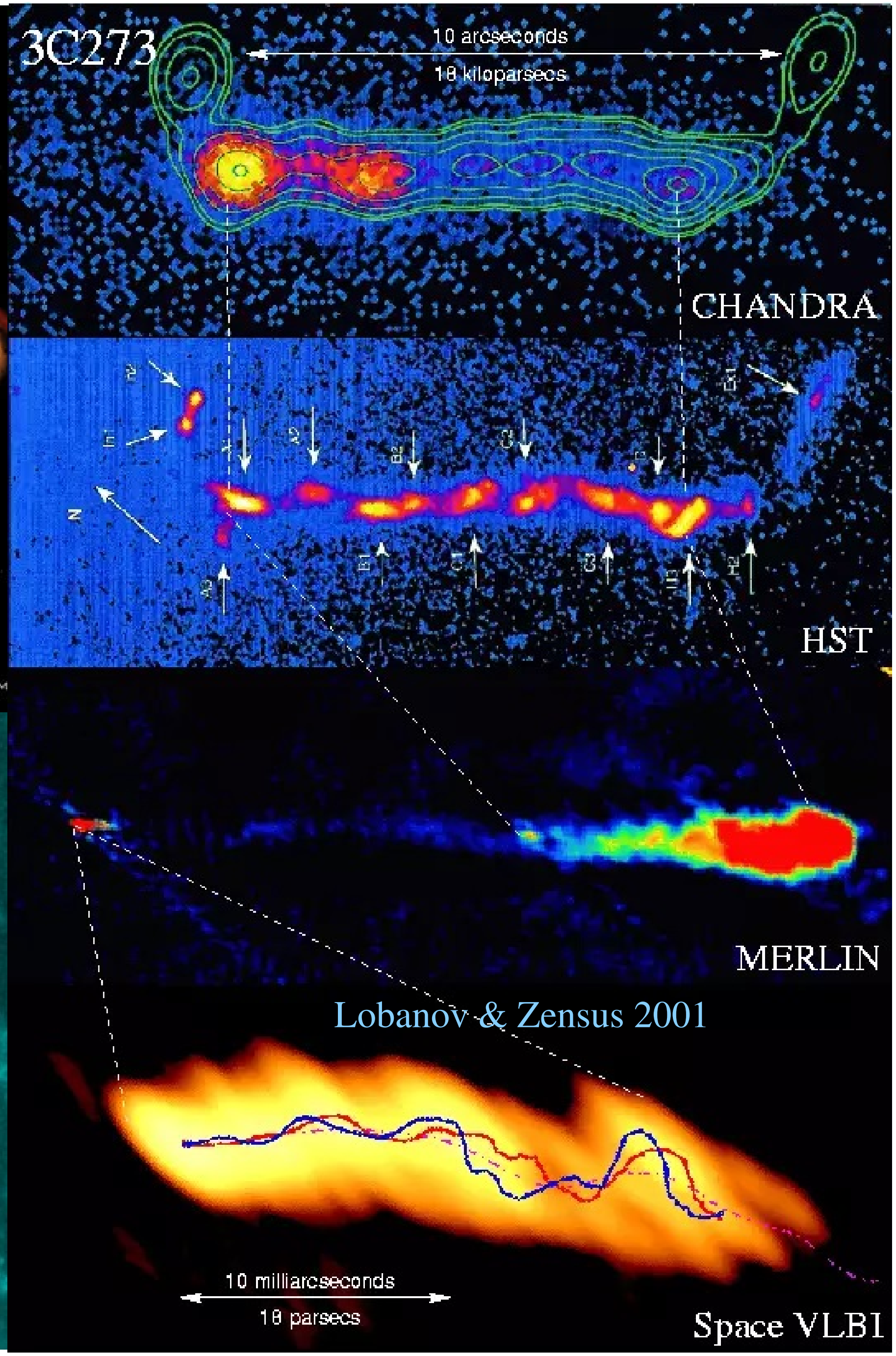
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10 arcseconds
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How to study MHD instabilities

Sketch of the methodology of a linear analysis:

- choose appropriate theory (fluid approach including or not gravity, relativity, viscosity, resistivity, heat transfer, radiative cooling, . . .)

- define the steady unperturbed state, add perturbation and linearize the equations

- decomposition to normal modes – whenever possible – offers a way to investigate the physics underlying each mechanism and find the growth rates

In the following ideal MHD will be used, in planar or cylindrical coordinates. In each case a “principal” equation will be found, whose solution, subject to given boundary conditions, specify the dispersion relation

Ideal (non-relativistic) MHD

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{V}) = 0 \,,$$

$$\left(\frac{\partial}{\partial t} + \boldsymbol{V} \cdot \nabla\right) P = c_s^2 \left(\frac{\partial}{\partial t} + \boldsymbol{V} \cdot \nabla\right) \rho \,,$$

$$\rho \left(\frac{\partial}{\partial t} + \boldsymbol{V} \cdot \nabla\right) \boldsymbol{V} = -\nabla P + (\nabla \times \boldsymbol{B}) \times \boldsymbol{B} + \rho \boldsymbol{g} \,,$$

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{V} \times \boldsymbol{B}) \,,$$

$$\nabla \cdot \boldsymbol{B} = 0 \,.$$

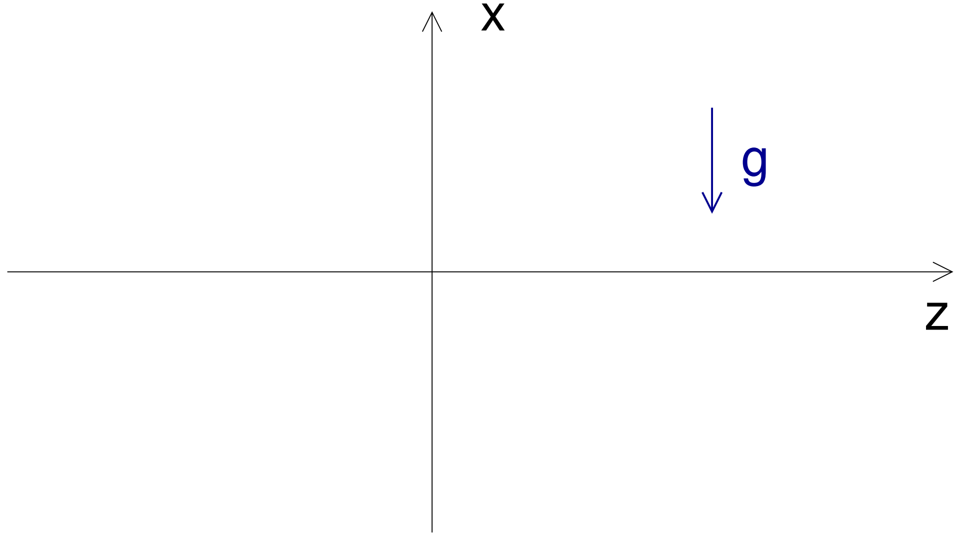
(Lorentz-Heaviside units)

Introduce the total pressure

$$\Pi = P + \frac{B^2}{2} \,.$$

Example in planar geometry

- Suppose the unperturbed steady-state has density $\rho_0(x)$, pressure $P(x)$, bulk velocity $\boldsymbol{V}_0 = V_{0z}(x)\hat{z} + V_{0y}(x)\hat{y}$, magnetic field $\boldsymbol{B}_0 = B_{0z}(x)\hat{z} + B_{0y}(x)\hat{y}$, and the gravity is uniform $\boldsymbol{g} = -g\hat{x}$.



(e.g. two semi-infinite plasmas in contact at the interface $x = 0$)

- The zeroth order equations are satisfied provided that

$$\frac{d\Pi_0}{dx} = -\rho_0 g\,.$$

Linearization

- Adding perturbation $\rho = \rho_0(x) + \rho_1(\boldsymbol{r}, t)$, $\Pi = \Pi_0(x) + \Pi_1(\boldsymbol{r}, t)$, $\boldsymbol{V} = \boldsymbol{V}_0 + \boldsymbol{V}_1(\boldsymbol{r}, t)$, $\boldsymbol{B} = \boldsymbol{B}_0 + \boldsymbol{B}_1(\boldsymbol{r}, t)$, and linearizing we get differential equations whose coefficients depend only on x.

- This allows for Fourier decomposition into modes

$$\rho_1 = \rho_1(x) e^{i(k_y y + k_z z - \omega t)} \,,$$

$$\Pi_1 = \Pi_1(x) e^{i(k_y y + k_z z - \omega t)} \,,$$

$$\boldsymbol{V}_1 = \left[V_{1x}(x)\hat{x} + V_{1y}(x)\hat{y} + V_{1z}(x)\hat{z}\right] e^{i(k_y y + k_z z - \omega t)} \,,$$

$$\boldsymbol{B}_1 = \left[B_{1x}(x)\hat{x} + B_{1y}(x)\hat{y} + B_{1z}(x)\hat{z}\right] e^{i(k_y y + k_z z - \omega t)} \,.$$

The system becomes

$$\left(\begin{array}{c} 8 \times 10 \text{ array} \\ \text{function of } x \text{ (and } \omega\,, k_y\,, k_z\text{)} \end{array}\right) \left(\begin{array}{c} \rho_1 \\ B_{1x} \\ B_{1y} \\ B_{1z} \\ V_{1y} \\ V_{1z} \\ dV_{1x}/dx \\ d\Pi_1/dx \\ V_{1x} \\ \Pi_1 \end{array}\right) = 0$$

and gives ρ_1, B_{1x}, B_{1y}, B_{1z}, V_{1y}, V_{1z}, $\dfrac{dV_{1x}}{dx}$, $\dfrac{d\Pi_1}{dx}$ as functions of V_{1x}, Π_1.

Thus the system is reduced to the two equations

$$\frac{d}{dx}\begin{pmatrix} V_{1x} \\ \Pi_1 \end{pmatrix} + \begin{pmatrix} 2\times 2 \text{ array} \\ \text{function of } x \text{ (and } \omega\,, k_y\,, k_z\text{)} \end{pmatrix}\begin{pmatrix} V_{1x} \\ \Pi_1 \end{pmatrix} = 0\,.$$

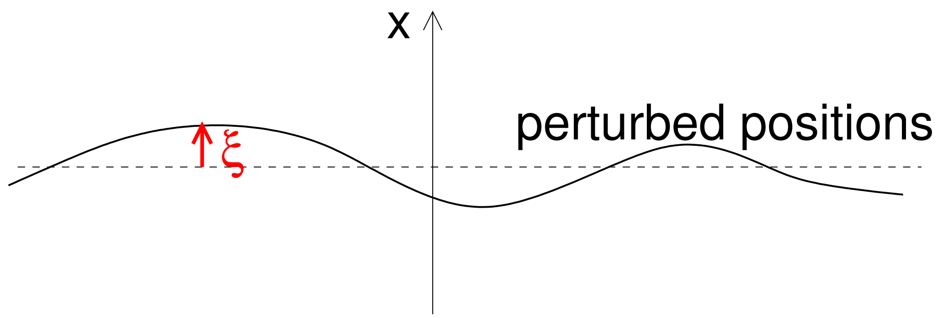
Better to use the Lagrangian displacement ξ in the x direction, connected with the velocity

$$V_{1x} = \frac{d\xi}{dt} \approx \frac{\partial \xi}{\partial t} + \boldsymbol{V}_0 \cdot \nabla \xi$$

It has the form $\xi = \xi(x) e^{i(k_y y + k_z z - \omega t)}$, so

$$V_{1x} \approx \frac{\partial \xi}{\partial t} + \boldsymbol{V}_0 \cdot \nabla \xi = -i\omega_0 \xi\,, \quad \omega_0 = \omega - \boldsymbol{k}_0 \cdot \boldsymbol{V}_0\,, \quad \boldsymbol{k}_0 = k_y \hat{y} + k_z \hat{z}$$

Two important quantities



Instead of V_{1x} and Π_1 use

- the Lagrangian displacement $y_1 \equiv \xi$, and
- the perturbation of the total pressure in the perturbed position

$$\Pi_1 + \xi \frac{d\Pi}{dx} \approx \Pi_1 + \xi \frac{d\Pi_0}{dx}, \text{ or, } y_2 \equiv \Pi_1 + y_1 \frac{d\Pi_0}{dx}$$

The advantage is that these two quantities are everywhere continuous, even at locations where the unperturbed state has contact discontinuities

The system becomes

$$\frac{d}{dx}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = 0\,,$$

where $\quad f_{11} = g\rho_0\left(\frac{\tilde{\kappa}^2}{A} - \frac{\rho_0\omega_0^2}{S}\right)\,, \quad f_{12} = \frac{\tilde{\kappa}^2}{A}\,,$

$$f_{21} = -A - g^2\rho_0^2\left(\frac{\tilde{\kappa}^2}{A} - \frac{\rho_0\omega_0^2 + F^2}{S}\right)\,, \quad f_{22} = -f_{11}\,,$$

$$F = \boldsymbol{k}_0\cdot\boldsymbol{B}_0\,, \quad A = \rho_0\left(\omega_0^2 - \frac{F^2}{\rho_0}\right)\,, \quad S = \rho_0^2\left[\left(\frac{B_0^2}{\rho_0} + c_s^2\right)\omega_0^2 - \frac{F^2}{\rho_0}c_s^2\right]\,,$$

$$\tilde{\kappa}^2 = \frac{\rho_0^2\omega_0^4}{S} - k_0^2\,.$$

All other perturbations are given functions of y_1, y_2, e.g.,
$\rho_1 + y_1\rho_0' = \frac{\rho_0^2}{S}(F^2 y_1 + \omega_0^2 y_2)$ and incompressibility means $S \to \infty$.

The “principal” equation

Since the system is linear only the ratio $Y = \dfrac{y_1}{y_2}$ is uniquely defined. To solve the equation for this (complex) function is sufficient (minimalist approach)

$$\boxed{\frac{dY}{dx} = f_{21}Y^2 + (f_{22} - f_{11})Y - f_{12}}$$

We just need to integrate this single, first order, differential equation, requiring Y to be everywhere continuous and satisfying the correct boundary conditions at the extreme values of x.

Knowing Y we can find y_1, y_2 from

$$\frac{y_2'}{y_2} = -f_{21}Y - f_{22}\,, \qquad y_1 = Y y_2 \qquad \text{(or } \frac{y_1'}{y_1} = -f_{11} - f_{12}\frac{1}{Y}\text{)}$$

- The goal is to find the dispersion relation between ω and $\boldsymbol{k}_0$, such that Y satisfies the correct boundary conditions at the two extreme values of x (this is an eigenvalue problem like Sturm-Liouville in quantum mechanics)

- Temporal approach will be followed, i.e., give real k_y, k_z and find complex ω for which the principal equation satisfies the boundary conditions. Since $e^{i(k_y y+k_z z-\omega t)} = e^{\Im\omega t} e^{i(k_y y+k_z z-\Re\omega t)}$ the $\Im\omega$ represents the growth rate of the mode (if positive)

- (Spatial approach is also possible: give real ω, k_y and find complex k_z, in which case $e^{i(k_y y+k_z z-\omega t)} = e^{-\Im k_z z} e^{i(k_y y+\Re k_z z-\omega t)}$ and $1/|\Im k_z|$ represents the growth length)

Boundary conditions

- At plasma interfaces Y is continuous

- If plasma has solid boundary(ies), Y vanishes there

- At $x = +\infty$: Near infinity the plasma is uniform and Y approaches a constant, given by the principal equation

$$0 = f_{21}Y^2 + 2f_{22}Y - f_{12} \Leftrightarrow Y = \frac{-f_{22} \pm \sqrt{f_{22}^2 + f_{12}f_{21}}}{f_{21}}.$$

The correct sign corresponds to decreasing $|y_1|$ and $|y_2|$, i.e., to $\dfrac{y_2'}{y_2} = -f_{21}Y - f_{22}$ with negative real part[1]

- Similarly at $x = -\infty$ the correct sign corresponds to $\dfrac{y_2'}{y_2} = -f_{21}Y - f_{22}$ with positive real part

[1] There is a way to find Y with the correct sign automatically (without the need to look at the equation for y_2), following the “Schwarzian approach”, for details see Vlahakis 2024

Methodology

- Shooting method:

Give $\boldsymbol{k}_0$ (k_y and k_z).

Give a trial value of (complex) ω.

Start the integration of the principal equation from the one end x_1, knowing the boundary value $Y|_{x=x_1}$, up to the other end x_2.

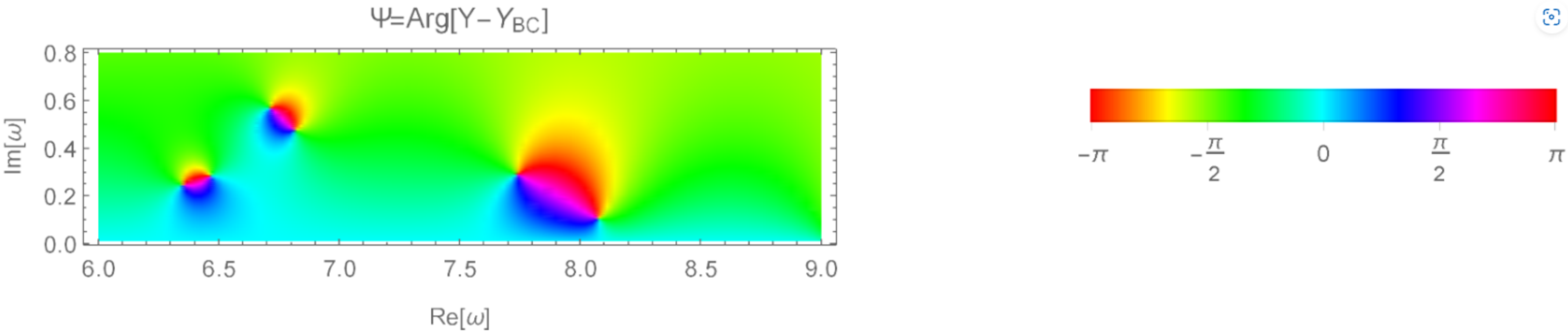
Check if at x_2 the integration gives the correct (known) boundary value Y_{BC}. If not change the trial value of ω and repeat.
The accepted ω are the roots of the (complex) dispersion relation

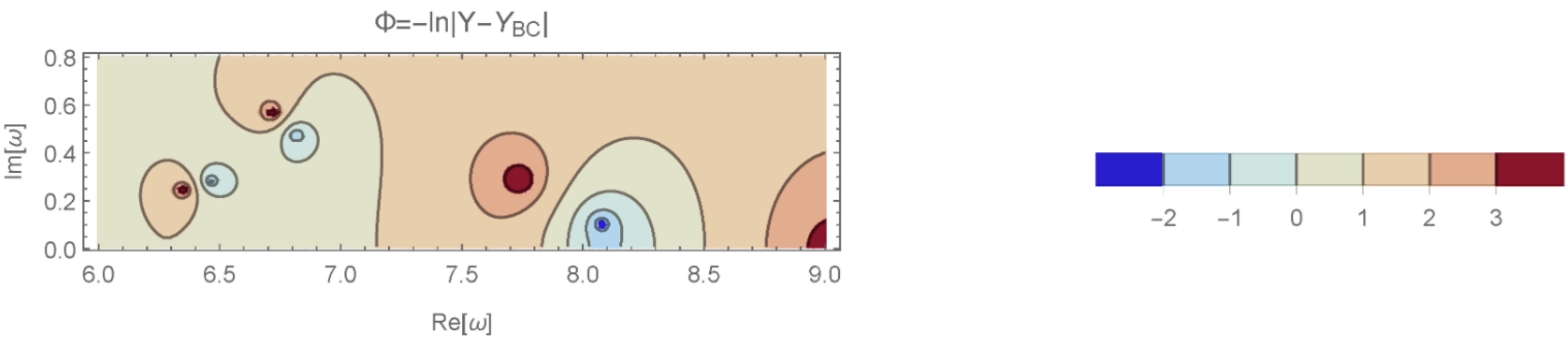
$$Y|_{x=x_2} - Y_{BC} = 0\,.$$

(In practice locate the roots by defining a grid in the complex ω plane and perform integration for each point.)

This way we find all accepted ω for which both boundary conditions are satisfied (for each $\boldsymbol{k}_0$).

Example





To find the roots it is sufficient to plot the isocontours of $\text{Arg}[Y|_{x=x_2} - Y_{BC}]$ in the ω plane
(details in the "minimalist approach", Vlahakis 2024)

Analytical results

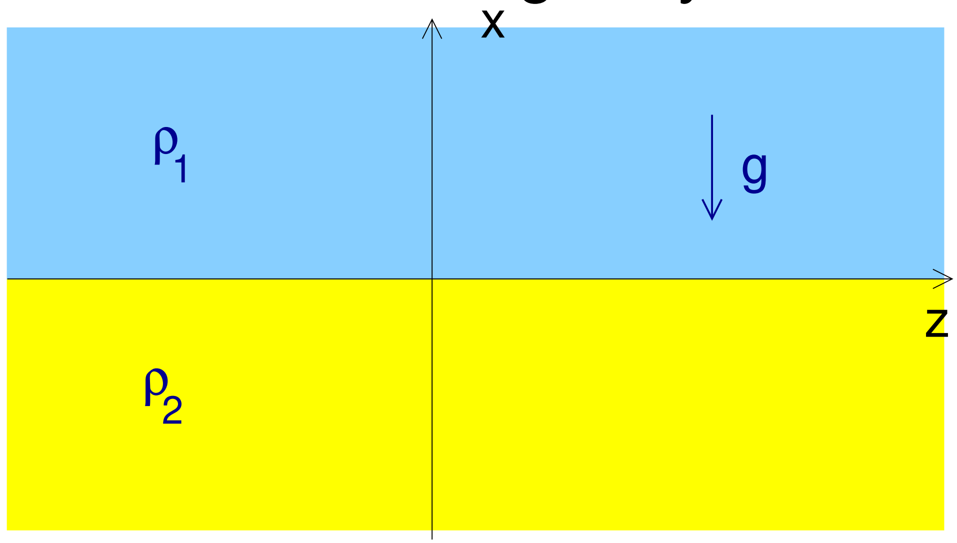
- In cases with simplified unperturbed states there are analytical solutions of the principal equation, leading to analytical expressions of the dispersion relation.

These offer important insight into the physics of the various instabilities.

Next we explore some of these classical cases.

Two semi-infinite incompressible plasmas

Two semi-infinite uniform and static plasmas with contact discontinuity at $x = 0$, inside zero gravity



Incompressibility: $S \to \infty$.

$$f_{11} = 0,\ f_{12} = -\frac{k_0^2}{A},\ f_{21} = -A,\ f_{22} = 0$$

The principal equation is $\dfrac{dY}{dx} = \dfrac{k_0^2}{A} - AY^2$ and has constant solutions $Y = \mp\dfrac{k_0}{A}$ in the two parts.

Checking the sign of $\dfrac{y_2'}{y_2} = -f_{21}Y - f_{22} = \mp k_0$ we deduce that the upper sign should be used for $x > 0$ and the lower for $x < 0$.

Thus $Y = -\dfrac{k_0}{\rho_1\omega^2 - F_1^2}$ for $x > 0$ and $Y = +\dfrac{k_0}{\rho_2\omega^2 - F_2^2}$ for $x < 0$.

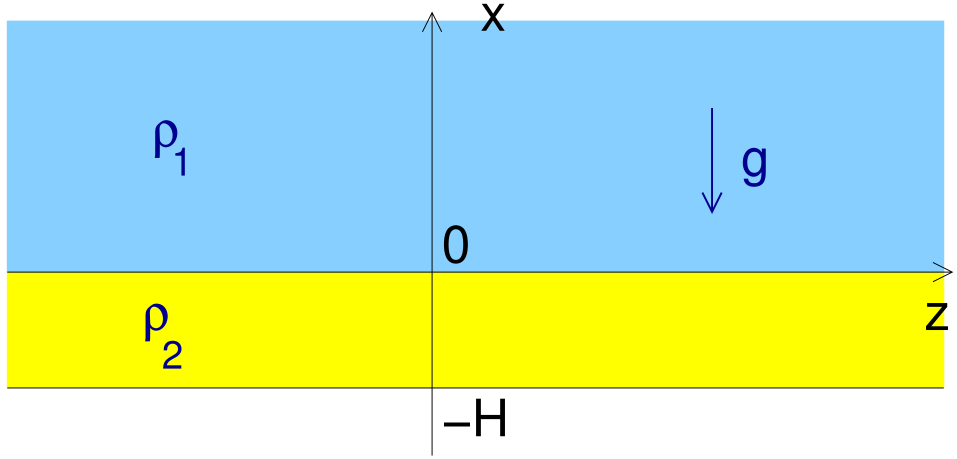
The continuity of Y gives the dispersion relation

$$\omega^2 = \frac{F_1^2 + F_2^2}{\rho_1 + \rho_2}$$

representing stable Alfvén waves. (We remind $F_1 = \boldsymbol{k}_0 \cdot \boldsymbol{B}_{01}$ and $F_2 = \boldsymbol{k}_0 \cdot \boldsymbol{B}_{02}$.)

Effect of finite depth

Suppose the bottom plasma has finite depth H.

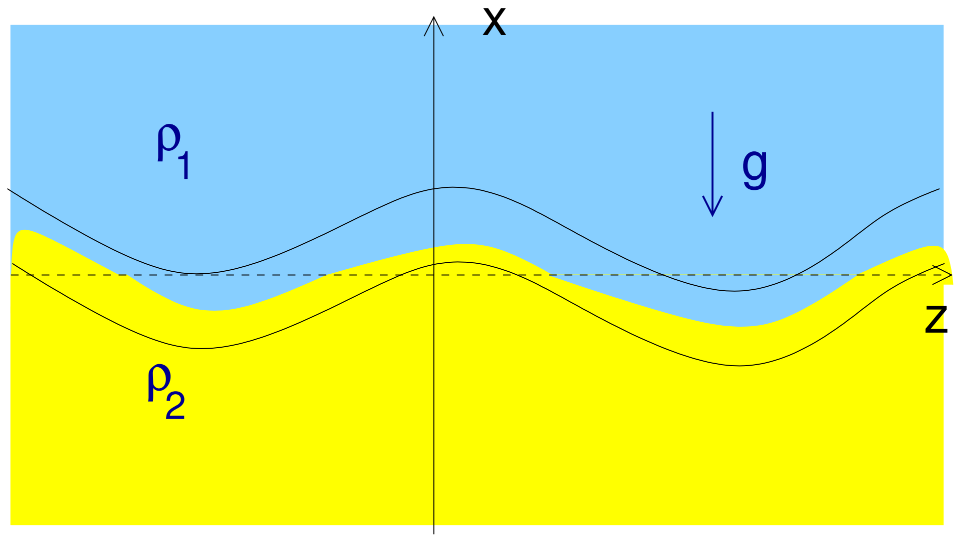


Then in the bottom part the solution of the principal equation that vanishes at $x = -H$ is $Y = \frac{k_0}{A} \tanh[k_0(x + H)]$.

The continuity of Y at $x = 0$ gives the dispersion relation

$$\omega^2 = \frac{F_1^2 + F_2^2 \coth(k_0 H)}{\rho_1 + \rho_2 \coth(k_0 H)}$$

The meaning of $F = \boldsymbol{k}_0 \cdot \boldsymbol{B}_0$



In general F represents the action of the magnetic tension which acts like a spring

The restoring force per mass is

$$-\omega^2 \xi = -k^2 v_A^2 \xi \,, \quad v_A = \frac{F}{\sqrt{\rho_0}}$$

(using the dispersion relation of the classical Alfvén waves)

Add gravity (Rayleigh-Taylor instability)

$$f_{11} = -g\rho_0 \frac{k_0^2}{A},\ f_{12} = -\frac{k_0^2}{A},\ f_{21} = -A + g^2\rho_0^2 \frac{k_0^2}{A},\ f_{22} = -f_{11}$$

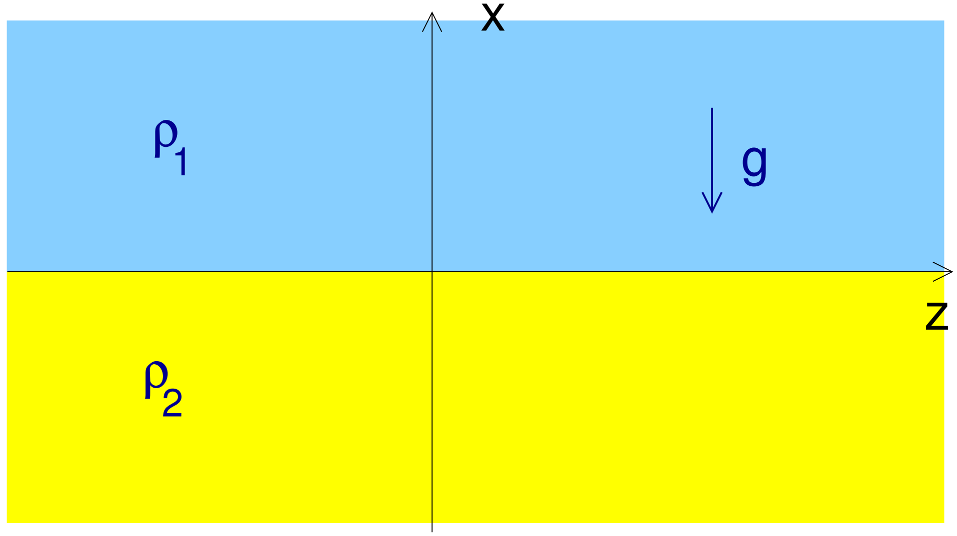
The principal equation is $\dfrac{dY}{dx} = \dfrac{k_0^2(1 + g\rho_0 Y)^2}{A} - AY^2$ and has constant solutions $Y = \dfrac{-k_0}{\pm A + \rho_0 g k_0}$ in the two semi-infinite parts.

Checking the sign of $\dfrac{y_2'}{y_2} = -f_{21}Y - f_{22} = \mp k_0$ we find

$Y = \dfrac{-k_0}{\rho_1\omega^2 - F_1^2 + \rho_1 g k_0}$ for $x > 0$ and $Y = \dfrac{-k_0}{-\rho_2\omega^2 + F_2^2 + \rho_2 g k_0}$ for $x < 0$.

The continuity of Y gives the dispersion relation

$$\omega^2 = -\frac{\rho_1 - \rho_2}{\rho_1 + \rho_2} g k_0 + \frac{F_1^2 + F_2^2}{\rho_1 + \rho_2}$$



For example, suppose we have a heavy unmagnetized plasma on top of a much lighter plasma whose magnetic field provides support. Then

$$\omega^2 = -gk_0 + \frac{(\boldsymbol{k}_0 \cdot \boldsymbol{B}_{02})^2}{\rho_1}$$

and the magnetic field stabilizes the Rayleigh-Taylor instability if it is sufficiently strong and the angle between the wavenumber and the field is small enough.

Add velocity (Kelvin-Helmholtz instability)

$$f_{11} = -g\rho_0 \frac{k_0^2}{A},\ f_{12} = -\frac{k_0^2}{A},\ f_{21} = -A + g^2\rho_0^2 \frac{k_0^2}{A},\ f_{22} = -f_{11}$$

Principal equation: $\dfrac{dY}{dx} = \dfrac{k_0^2(1 + g\rho_0 Y)^2}{\rho_0\omega_0^2 - F^2} - (\rho_0\omega_0^2 - F^2)Y^2$

Similarly we find dispersion relation

$$\rho_1(\omega - \boldsymbol{k}_0 \cdot \boldsymbol{V}_{01})^2 + \rho_2(\omega - \boldsymbol{k}_0 \cdot \boldsymbol{V}_{02})^2 = F_1^2 + F_2^2 - (\rho_1 - \rho_2) g k_0$$

with solutions

$$\omega = \frac{\rho_1 \boldsymbol{V}_{01} + \rho_2 \boldsymbol{V}_{02}}{\rho_1 + \rho_2} \cdot \boldsymbol{k}_0 \pm i \sqrt{\frac{\rho_1\rho_2 \left[(\boldsymbol{V}_{01} - \boldsymbol{V}_{02}) \cdot \boldsymbol{k}_0\right]^2}{(\rho_1 + \rho_2)^2} + \frac{\rho_1 - \rho_2}{\rho_1 + \rho_2} g k_0 - \frac{F_1^2 + F_2^2}{\rho_1 + \rho_2}}$$

Evidently the magnetic field has a stabilizing effect on the Kelvin-Helmholtz instability

Effect of compressibility

For simplicity ignore gravity.

Then $f_{11} = 0$, $f_{12} = \dfrac{\tilde{\kappa}^2}{A}$, $f_{21} = -A$, $f_{22} = 0$, and

$\tilde{\kappa}^2 = \dfrac{\rho_0^2 \omega_0^4}{S} - k_0^2 \Leftrightarrow \tilde{\kappa} = \Re\tilde{\kappa} + i\Im\tilde{\kappa}$ with $\Im\tilde{\kappa} > 0$ (arbitrarily chosen).

The solution of the principal equation $\dfrac{dY}{dx} = -AY^2 - \dfrac{\tilde{\kappa}^2}{A}$ is

$Y = \pm i\dfrac{\tilde{\kappa}}{A}$ with the upper sign for $x > 0$ and the lower for $x < 0$

(the $\dfrac{y_2'}{y_2} = \pm i\tilde{\kappa} \Leftrightarrow y_2 \propto e^{\mp\Im\tilde{\kappa}x} e^{\pm i\Re\tilde{\kappa}x}$, so $1/\Im\tilde{\kappa}$ is the decay length and $\pm\Re\tilde{\kappa}$ is the wavenumber in the $\hat{x}$ direction)

The dispersion relation is $\dfrac{\tilde{\kappa}_1}{A_1} = -\dfrac{\tilde{\kappa}_2}{A_2}$

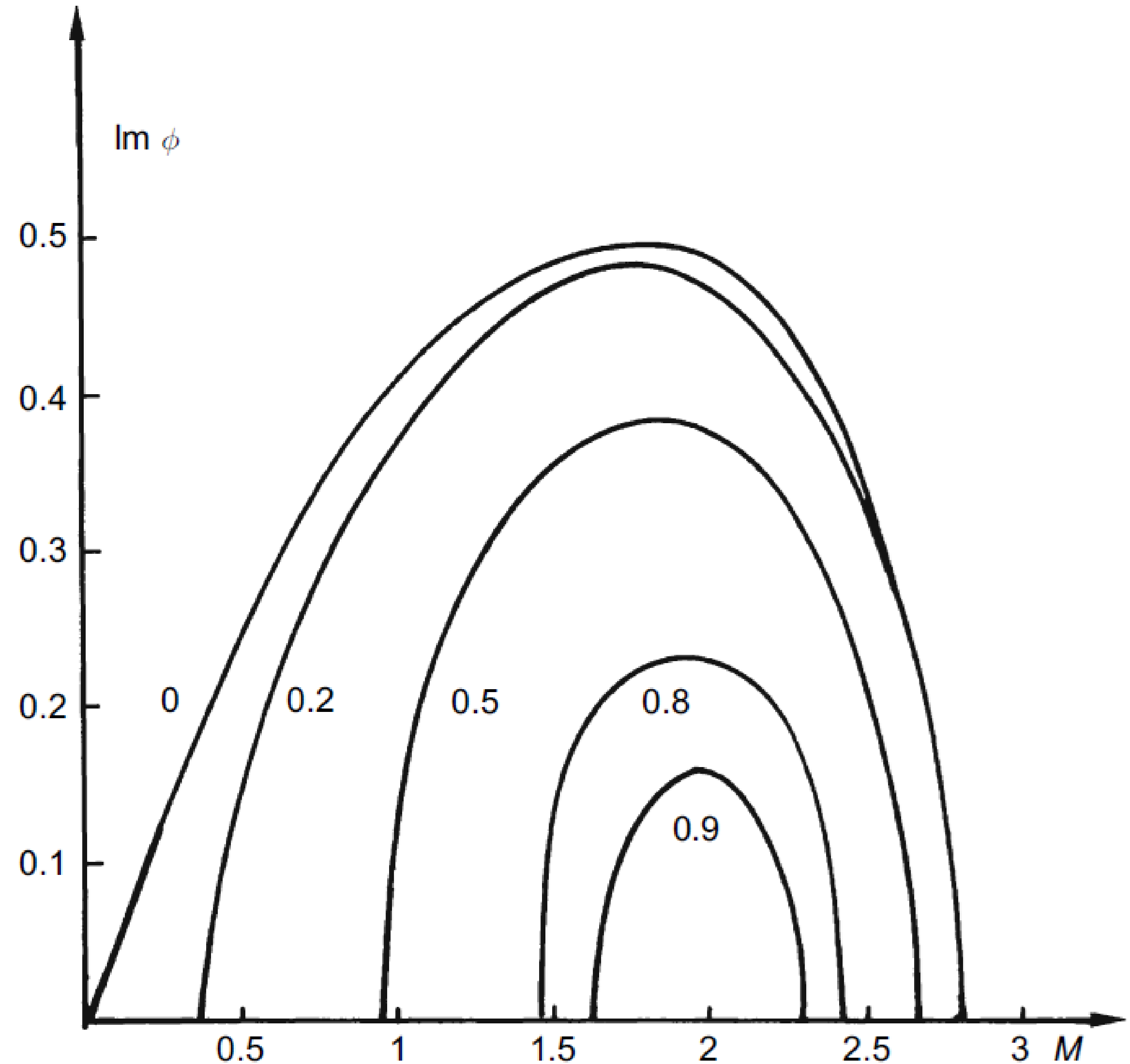
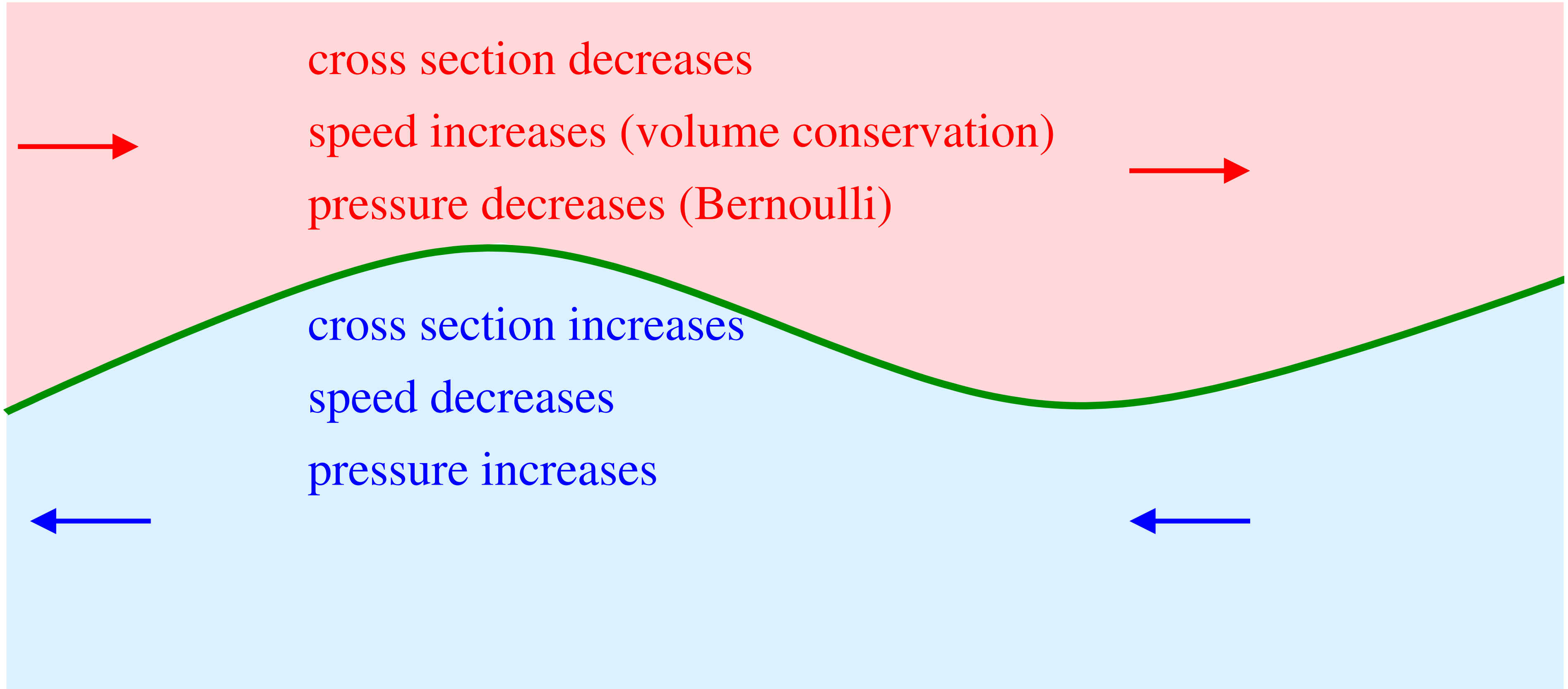


Fig. 2. Plot of the nondimensional growth rate Im Φ vs M of a perturbation propagating along the fluid velocity for a vortex sheet configuration, and for unmagnetized and magnetized flows (labels are different values of v_A/s)

Typical result for $\boldsymbol{k}_0 \parallel \boldsymbol{B}_0$, from Trussoni 2009 (review on KHI)

The physics of KHI

- Simplest variant: incompressible hydrodynamics



Always unstable

$$\omega = \frac{\rho_1 \boldsymbol{V}_{01} + \rho_2 \boldsymbol{V}_{02}}{\rho_1 + \rho_2} \cdot \boldsymbol{k}_0 \pm i \frac{\sqrt{\rho_1 \rho_2}}{\rho_1 + \rho_2} (\boldsymbol{V}_{01} - \boldsymbol{V}_{02}) \cdot \boldsymbol{k}_0$$

Compressibility effects

- Fast hydrodynamic flows (with sufficiently high Mach numbers) are stable

The Venturi tube becomes de Laval nozzle

When the section increases the density decreases – and not the speed as in the incompressible limit – consequently pressure decreases

(Time dependence is also important)

For example, for same fluids without magnetic field, stability when

$$M = \frac{|V_{01} - V_{02}|}{c_s} > \sqrt{8}$$

- Magnetic tension stabilizes

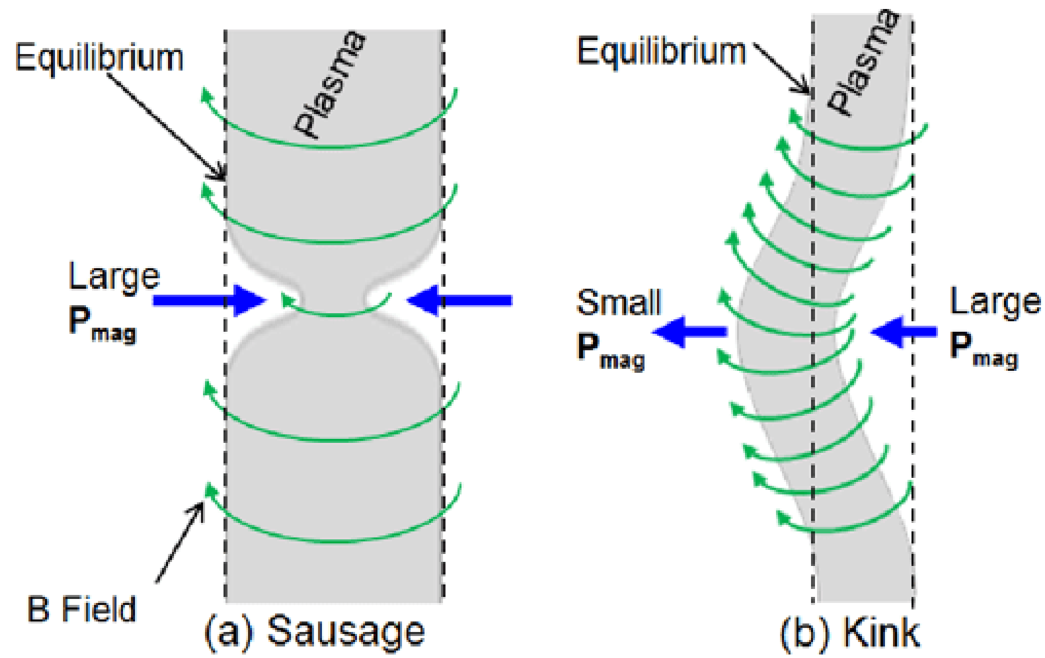
- Magnetic pressure destabilizes (the Venturi tube becomes de Laval nozzle if the flow approaches the fast magnetosonic speeds and not the sound speed)

Study in cylindrical geometry

Important for the stability of astrophysical jets

- contrary to planar geometry: (i) the jet is inhomogeneous (unavoidable if there is $B_{0\phi}$), (ii) there is a spatial scale r_j, (iii) reflections from the symmetry axis may be important

- Besides KHI there are also current-driven instabilities (sketch from Yager-Elorriaga 2017)



Example in cylindrical geometry

- For simplicity ignore here gravity and relativity
- Unperturbed cylindrical jet:

helical, axisymmetric, cylindrically symmetric and steady flow

$$\boldsymbol{V}_0 = V_{0z}(r)\hat{z} + V_{0\phi}(r)\hat{\phi}\,,$$

$$\boldsymbol{B}_0 = B_{0z}(r)\hat{z} + B_{0\phi}(r)\hat{\phi}\,,$$

$$\rho_0 = \rho_0(r)\,, \quad \Pi_0 = \Pi_0(r)\,.$$

Equilibrium condition $\displaystyle \frac{d\Pi_0}{dr} + \frac{B_{0\phi}^2}{r} - \rho_0 \frac{V_{0\phi}^2}{r} = 0\,.$

Linearized equations

With expansion to normal modes (with integer m)

$$Q(r\,,z\,,\phi\,,t) = Q_0(r) + Q_1(r)e^{i(m\phi+kz-\omega t)}$$

the equations again reduce to

$$\frac{d}{dr}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = 0\,,$$

where $\quad y_1 = r\xi = i\frac{rV_{1r}}{\omega_0}\,, \qquad y_2 = \Pi_1 + \frac{y_1}{r}\frac{d\Pi_0}{dr}\,.$

The y_1 and y_2 are everywhere continuous, the same for their ratio $Y = \frac{y_1}{y_2}$ that satisfies the principal equation (same as before, although the expressions of f_{ij} are more complicated)

Boundary conditions on the axis

- the functions $V_{0\phi}/r$, $B_{0\phi}/r$ and their derivatives are finite at $r = 0$ (meaning that the angular velocity and the poloidal current are regular functions at $r = 0$)

- For $m \neq 0$ the solutions near the axis behave as

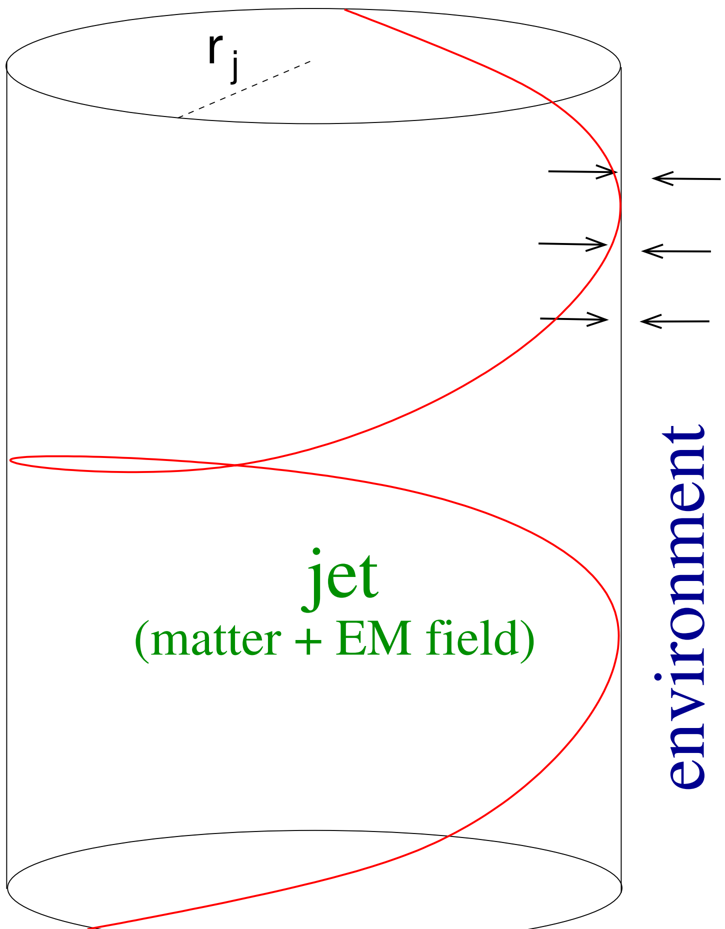
$$y_1 \propto r^{|m|}\,, \quad y_2 \propto r^{|m|}\,,$$

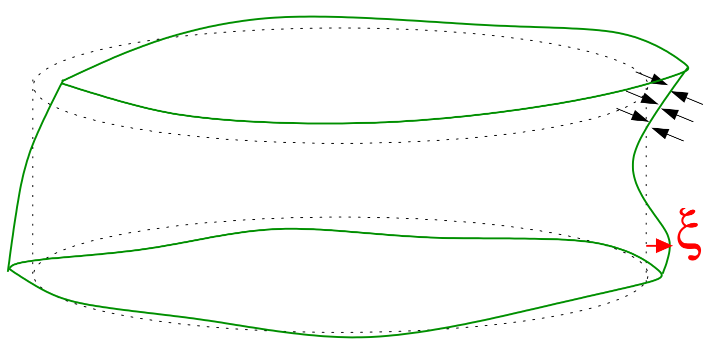
with fixed constant ratio $\quad Y(0) = \lim_{r \to 0} \frac{r f_{11} - |m|}{r f_{21}}\,.$

Since $\xi = \frac{y_1}{r}$ only the modes $m = \pm 1$ (kink) displace the jet axis

- For $m = 0$ the solutions near the axis behave as

$$y_2 = \text{const}\,, \quad y_1 = -\frac{1}{2} r^2 y_2 \lim_{r \to 0} \frac{f_{12}}{r}\,, \quad \text{so} \quad Y(0) = 0\,.$$





Eigenvalue problem

- integrate the principal equation inside the jet, starting from the axis (very rare to have analytical solution)
- continue in the environment
- the condition the perturbation to vanish at $r \to \infty$ gives the dispersion relation
- in many cases there are analytical solutions for the environment (expressed through Bessel functions) – in that case the matching of solutions at r_j gives the dispersion relation
- same procedure if there are more than one contact discontinuities

A cylindrical plasma column

- Static column with constant $\boldsymbol{B}_0 = B_0\hat{z}$ surrounded by vacuum

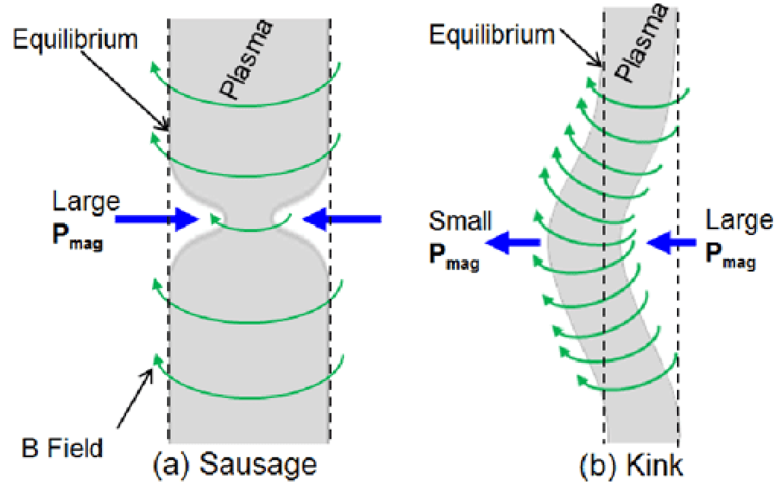
- The external force-free field is $\boldsymbol{B}_0 = B_\phi(r)\hat{\phi} + B_z\hat{z}$, with $B_\phi(r) \propto 1/r$ and constant B_z

- Assuming incompressibility we can find analytical expressions and the dispersion relation is

$$\omega^2 = \frac{k^2 B_0^2}{\rho} + \frac{[kB_z + \frac{m}{R}B_\phi(R)]^2}{\rho}\frac{I_m'(kR)K_m(kR)}{-K_m'(kR)I_m(kR)} - \frac{k^2[B_\phi(R)]^2}{\rho}\frac{I_m'(kR)}{kRI_m(kR)}$$

- The first two terms are positive (tension) while the last is negative (pressure due to B_ϕ)

• Strong enough B_ϕ leads to instability. For $m = 1$ and $kR \ll 1$, $k > 2\pi/L$, where L the length of the column, we arrive at the Kruskal-Shafranov stability criterion $|B_\phi/B_z| < 2\pi R/L$



• ω is either real (the column is stable) or purely imaginary ($\Re\omega = 0$, so growing perturbation without oscillations)

• The ω^2 is minimum (maximum growth rate $\Im\omega$) around the "resonance" $kB_z + \frac{m}{R}B_\phi(R) = 0$ (this is $\boldsymbol{k}_0 \cdot \boldsymbol{B}_0|_{r=R} = 0$ since $\boldsymbol{k}_0 = k\hat{z} + \frac{m}{r}\hat{\phi}$)

• The above characterize the "current-driven instabilities" (if the column was moving $\Re\omega_0 = 0$ meaning that the perturbation is advected with the flow)

static column, from Appl, Lery & Baty 2000 (pitch $P \equiv rB_z/B_\phi$)

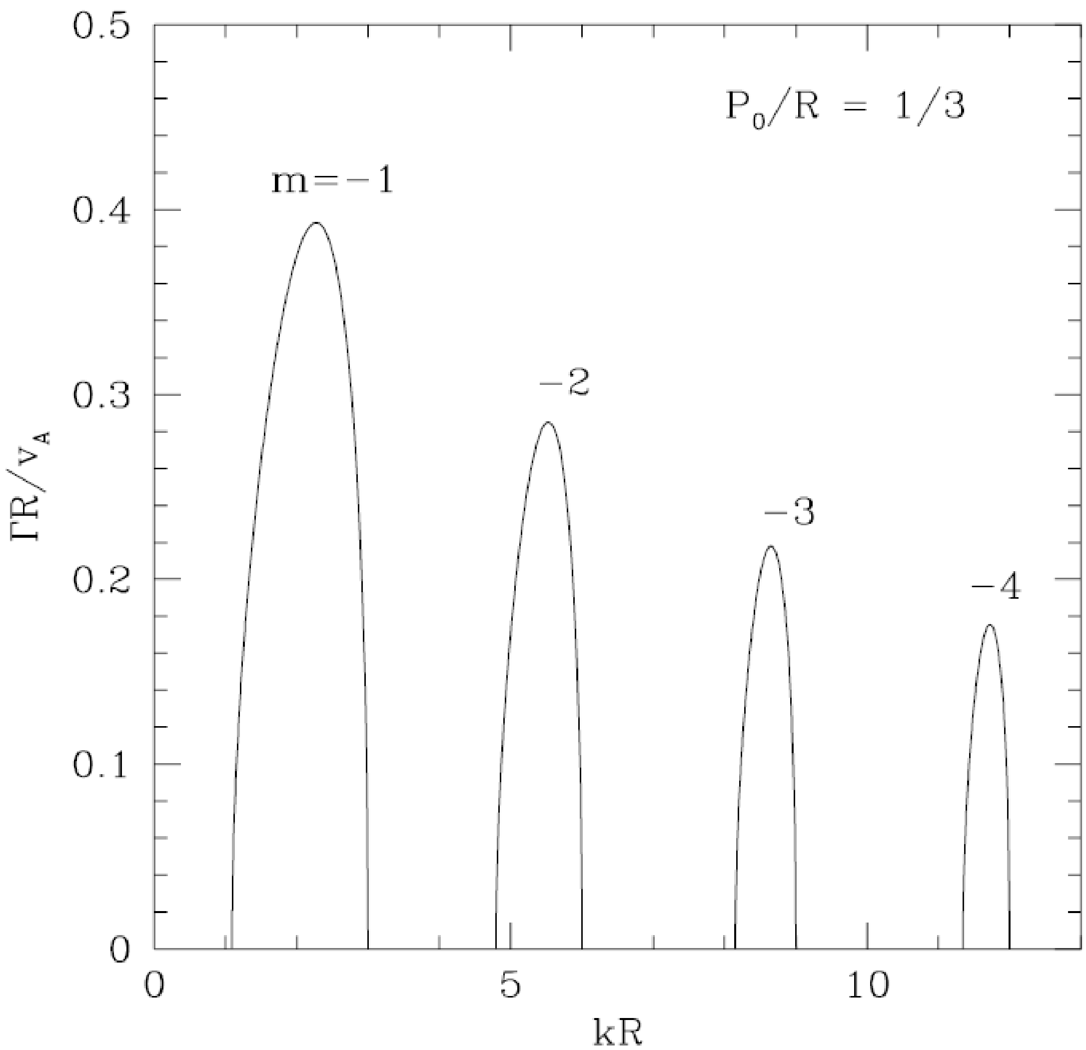
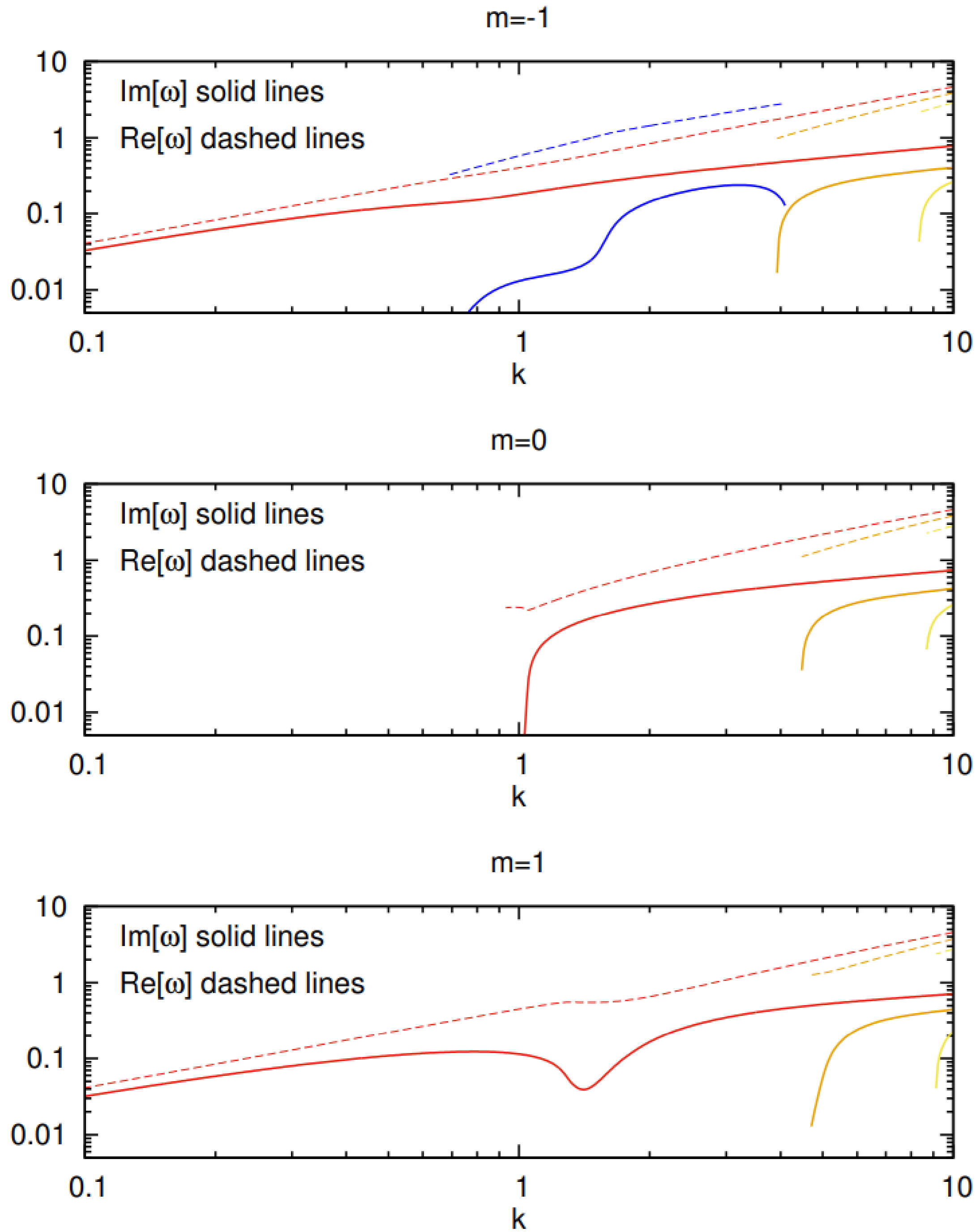
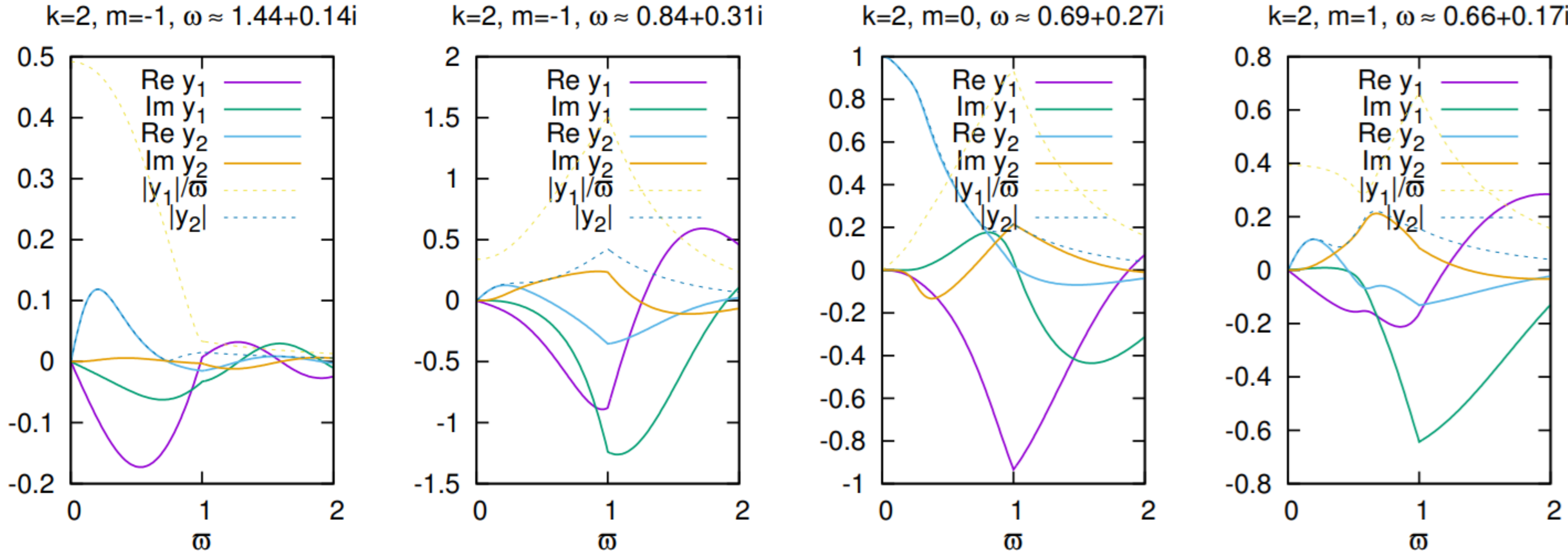


Fig. 1. Constant pitch equilibrium with $P_0/R = \frac{1}{3}$. Growth rate for $m = -1, -2, -3, -4$ (from the left) current-driven instabilities. The instability sets in from the short wavelength side at the resonant condition $k_0 = -m/P_0$.

relativistic jet, from Vlahakis 2024





CDI is mostly concentrated near the axis
(and displaces the axis, as expected for $|m| = 1$).

It appears near resonance $\boldsymbol{k}_{\rm co} \cdot \boldsymbol{B}_{\rm co} = 0$ (which is possible only for $m < 0$ since $B_\phi > 0$ and $B_z > 0$).

It has $\Re\omega_{\rm co} \approx 0$ (advected with the flow).

KHI perturbation mostly near the jet boundary
(even more localized for higher k).

Pros and cons of normal mode analysis

Disadvantages:

- requires the unperturbed state to be one dimensional, either function of x in planar geometry, or function of r in cylindrical

- does not cover the nonlinear phase

Advantages:

- gives growth rates

- gives analytical expressions in certain limits
  in spite the simplifications these build our understanding
  (and can roughly be applied in more complicated cases)

Other ways to study instabilities

• Variation formulation and the energy principle:
Work with the Lagrangian displacement, and find the variation of a "potential energy". The various terms of this expression, when negative, correspond to instabilities. See e.g. the book by Boyd & Sanderson, or by Goedbloed, Keppens, & Poeds (references).
⋆ Advantage: helps to understand the instability mechanisms
⋆ Disadvantage: does not give growth rates

• Numerical solutions of the full problem (simulations, PINNs):
⋆ Advantages: It is the only way to study the nonlinear phase and works for any unperturbed state
⋆ Disadvantage: computationally expensive, hard to cover the parameter space and control numerical errors (unavoidable in the small wavelength limit)

The best strategy is always to combine all available methods.

Summary

⋆ Normal mode analysis and the "principal" equation is a powerful tool to study instabilities in magnetized plasmas, to explore the conditions that lead to instability and find the growth rates.

⋆ Magnetic tension stabilizes in all cases and corresponds to terms $F = \boldsymbol{k}_0 \cdot \boldsymbol{B}_0$ in the dispersion relation.
It's stabilization effect vanishes when $\boldsymbol{k}_0 \cdot \boldsymbol{B}_0 = 0$ though.

⋆ Magnetic pressure destabilizes in plasma columns.
Also in cases where compressibility is important (like in KHI), since it replaces the sound speed with the fast magnetosonic speed (so stability requires higher speeds compared to hydrodynamic cases).

⋆ Besides the KHI and CDI there are other instabilities not covered here, e.g. centrifugal, magnetorotational, . . .
(the presented formalism applies to these too)

References

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